Example 141. Find the unique solution $u(x,y)$ to:

$$u_{xx}+u_{yy}=0 \quad \text{(PDE)}$$
$$\begin{aligned} u(x,0) &= f(x) \\ u(x,b) &= 0 \\ u(0,y) &= 0 \\ u(a,y) &= 0 \end{aligned} \quad \text{(BC)}$$

Solution.

- We proceed as before and look for solutions $u(x,y)=X(x)Y(y)$ (**separation of variables**).
  Plugging into (PDE), we get $X''(x)Y(y)+X(x)Y''(y)$, and so $\frac{X''(x)}{X(x)}=-\frac{Y''(y)}{Y(y)}=\text{const}=:-\lambda$.
  We thus have $X''+\lambda X=0$ and $Y''-\lambda Y=0$.
- From the last three (BC), we get $X(0)=0$, $X(a)=0$, $Y(b)=0$.
  We ignore the first (inhomogeneous) condition for now.
- So X solves $X''+\lambda X=0$, $X(0)=0$, $X(a)=0$.
  From earlier, we know that, up to multiples, the only nonzero solutions of this eigenvalue problem are $X(x)=\sin\left(\frac{\pi n}{a}x\right)$ corresponding to $\lambda=\left(\frac{\pi n}{a}\right)^2$, $n=1,2,3\ldots$.
- On the other hand, Y solves $Y''-\lambda Y=0$, and hence $Y(y)=Ae^{\sqrt{\lambda}y}+Be^{-\sqrt{\lambda}y}$.
  The condition $Y(b)=0$ implies that $Ae^{\sqrt{\lambda}b}+Be^{-\sqrt{\lambda}b}=0$ so that $B=-Ae^{2\sqrt{\lambda}b}$.
  Hence, $Y(y)=A\left(e^{\sqrt{\lambda}y}-e^{\sqrt{\lambda}(2b-y)}\right)$.
- Taken together, we have the solutions $u_n(x,y)=\sin\left(\frac{\pi n}{a}x\right)\left(e^{\frac{\pi n}{a}y}-e^{\frac{\pi n}{a}(2b-y)}\right)$ solving (PDE)+(BC), with the exception of $u(x,0)=f(x)$.
- We wish to combine these in such a way that $u(x,0)=f(x)$ holds as well.
  At $y=0$, $u_n(x,0)=\sin\left(\frac{\pi n}{a}x\right)(1-e^{2\pi nb/a})$. All of these are $2a$-periodic.
  Hence, we extend $f(x)$, which is only given on $(0,a)$, to an odd $2a$-periodic function (its Fourier sine series!). By making it odd, its Fourier series will only involve sine terms: $f(x)=\sum_{n=1}^{\infty}b_n\sin\left(\frac{\pi n}{a}x\right)$.
  Note that
  $$b_n=\frac{1}{a}\int_{-a}^{a}f(x)\sin\left(\frac{n\pi x}{a}\right)\mathrm{d}x=\frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)\mathrm{d}x,$$
  where the first integral makes reference to the extension of $f(x)$ while the second integral only uses $f(x)$ on its original interval of definition.

Consequently, (PDE)+(BC) is solved by

$$u(x,y)=\sum_{n=1}^{\infty}\frac{b_n}{1-e^{2\pi nb/a}}u_n(x,y)=\sum_{n=1}^{\infty}\frac{b_n}{1-e^{2\pi nb/a}}\sin\left(\frac{\pi n}{a}x\right)\left(e^{\frac{\pi n}{a}y}-e^{\frac{\pi n}{a}(2b-y)}\right),$$

where

$$b_n=\frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)\mathrm{d}x.$$

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