Steady-state temperature

Review. (2D and 3D heat equation) In higher dimensions, the heat equation takes the form $u_t = k(u_{xx} + u_{yy})$ or $u_t = k(u_{xx} + u_{yy} + u_{zz})$.

Note that $\Delta u = u_{xx} + u_{yy} + u_{zz}$ is the Laplace operator you may know from Calculus III (more below).

If temperature is steady, then $u_t = 0$. Hence, the steady-state temperature $u(x, y)$ must satisfy the PDE $u_{xx} + u_{yy} = 0$.

(Laplace equation)

$$u_{xx} + u_{yy} = 0$$

Comment. The Laplace equation is so important that its solutions have their own name: **harmonic functions**.

Comment. Also known as the "potential equation"; satisfied by electric/gravitational potential functions.

Recall from Calculus III (if you have taken that class) that the gradient of a scalar function $f(x, y)$ is the vector field $\boldsymbol{F} = \operatorname{grad} f = \nabla f = \begin{bmatrix} f_x(x, y) \\ f_y(x, y) \end{bmatrix}$. One says that $\boldsymbol{F}$ is a **gradient field** and f is a **potential function** for $\boldsymbol{F}$ (for instance, $\boldsymbol{F}$ could be a gravitational field with gravitational potential f).

The divergence of a vector field $\boldsymbol{G} = \begin{bmatrix} g(x, y) \\ h(x, y) \end{bmatrix}$ is $\operatorname{div} \boldsymbol{G} = g_x + h_y$. One also writes $\operatorname{div} \boldsymbol{G} = \nabla \cdot \boldsymbol{G}$.

The gradient field of a scalar function f is divergence-free if and only if f satisfies the Laplace equation $\Delta f = 0$.

Other notations. $\Delta f = \operatorname{div} \operatorname{grad} f = \nabla \cdot \nabla f = \nabla^2 f$

Boundary conditions. For steady-state temperatures profiles, it is natural to prescribe the temperature on the boundary of a region $R \subseteq \mathbb{R}^2$ (or $R \subseteq \mathbb{R}^3$ in the 3D case).

Comment. Gravitational and electrostatic potentials (not in the vacuum) satisfy the **Poisson equation** $u_{xx} + u_{yy} = f(x, y)$, the inhomogeneous version of the Laplace equation.

(Dirichlet problem)

$u_{xx} + u_{yy} = 0$ within region R

$u(x, y) = f(x, y)$ on boundary of R

In general. A Dirichlet problem consists of a PDE, that needs to hold within a region R, and prescribed values on the boundary of that region ("Dirichlet boundary conditions").

In our next example we solve the Dirichlet problem in the case when R is a rectangle.

Important observation. We are using homogeneous boundary conditions for three of the sides. That is actually no loss of generality.

Indeed, note that in order to solve
$$\begin{array}{ll} u_{xx} + u_{yy} = 0 & \text{(PDE)} \\ u(x, 0) = f_1(x) & \\ u(x, b) = f_2(x) & \\ u(0, y) = f_3(y) & \text{(BC)} \\ u(a, y) = f_4(y) & \end{array}$$
we can solve the four Dirichlet problems:

$$\begin{array}{llll} u_{xx} + u_{yy} = 0 & u_{xx} + u_{yy} = 0 & u_{xx} + u_{yy} = 0 & u_{xx} + u_{yy} = 0 \\ u(x, 0) = f_1(x) & u(x, 0) = 0 & u(x, 0) = 0 & u(x, 0) = 0 \\ u(x, b) = 0 & u(x, b) = f_2(x) & u(x, b) = 0 & u(x, b) = 0 \\ u(0, y) = 0 & u(0, y) = 0 & u(0, y) = f_3(y) & u(0, y) = 0 \\ u(a, y) = 0 & u(a, y) = 0 & u(a, y) = 0 & u(a, y) = f_4(y) \end{array}$$

The sum of the four solutions then solves the Dirichlet problem we started with.

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