The inhomogeneous heat equation

We next indicate that we can similarly solve the inhomogeneous heat equation (with inhomogeneous boundary conditions).

Comment. We indicated earlier that

$$\begin{array}{ll} u_t = k u_{xx} & \text{(PDE)} \\ u(0,t) = a, \quad u(L,t) = b & \text{(BC)} \\ u(x,0) = f(x), \quad x \in (0,L) & \text{(IC)} \end{array}$$

can be solved by realizing that $Ax + B$ solves (PDE).

Indeed, let $v(x) = a + \frac{b-a}{L} x$ (so that $v(0) = a$ and $v(L) = b$). We then look for a solution of the form $u(x,t) = v(x) + w(x,t)$. Note that $u(x,t)$ solves (PDE)+(BC)+(IC) if and only if $w(x,t)$ solves:

$$\begin{array}{ll} w_t = k w_{xx} & \text{(PDE)} \\ w(0,t) = 0, \quad w(L,t) = 0 & \text{(BC*)} \\ w(x,0) = f(x) - v(x), \quad x \in (0,L) & \text{(IC)} \end{array}$$

This the (homogeneous) heat equation that we know how to solve.

$v(x)$ is called the **steady-state solution** (it does not depend on time!) and $w(x,t)$ the **transient solution** (note that $w(x,t)$ and its partial derivatives tend to zero as $t \to \infty$).

Example 139. Consider the heat flow problem: $\begin{array}{ll} u_t = 3u_{xx} + 4x^2 & \text{(PDE)} \\ u(0,t) = 1, \quad u_x(3,t) = -5 & \text{(BC)} \\ u(x,0) = f(x), \quad x \in (0,3) & \text{(IC)} \end{array}$

Determine the steady-state solution and spell out equations characterizing the transient solution.

Solution. We look for a solution of the form $u(x,t) = v(x) + w(x,t)$, where $v(x)$ is the steady-state solution and where $w(x,t)$ is the transient solution which (together with its derivatives) tends to zero as $t \to \infty$.

- Plugging into (PDE), we get $w_t = 3v'' + 3w_{xx} + 4x^2$. Letting $t \to \infty$, this becomes $0 = 3v'' + 4x^2$. Note that this also implies that $w_t = 3w_{xx}$.
- Plugging into (BC), we get $v(0) + w(0,t) = 1$ and $v'(3) + w_x(3,t) = -5$. Letting $t \to \infty$, these become $v(0) = 1$ and $v'(3) = -5$.
- Solving the ODE $0 = 3v'' + 4x^2$ with boundary conditions $v(0) = 1$ and $v'(3) = -5$, we find

  $$v(x) = \iint -\frac{4}{3} x^2 \mathrm{d}x \mathrm{d}x = -\frac{1}{9} x^4 + C_1 + C_2 x$$

  and therefore the steady-state solution $v(x) = -\frac{1}{9} x^4 + 1 + 7x$.

On the other hand, the transient solution $w(x,t)$ is characterized as the unique solution to:

$$\begin{array}{ll} w_t = 3w_{xx} & \text{(PDE*)} \\ w(0,t) = 0, \quad w_x(3,t) = 0 & \text{(BC*)} \\ w(x,0) = f(x) - v(x) & \text{(IC*)} \end{array}$$

This homogeneous heat flow problem can now be solved using separation of variables.

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Example 140. For $t \geqslant 0$ and $x \in [0, 4]$, consider the heat flow problem:
$$\begin{aligned} u_t &= 2u_{xx} + e^{-x/2} \\ u_x(0,t) &= 3 \\ u(4,t) &= -2 \\ u(x,0) &= f(x) \end{aligned}$$

Determine the steady-state solution and spell out equations characterizing the transient solution.

Solution. We look for a solution of the form $u(x,t) = v(x) + w(x,t)$, where $v(x)$ is the steady-state solution and where the transient solution $w(x,t)$ tends to zero as $t \to \infty$ (as do its derivatives).

- Plugging into (PDE), we get $w_t = 2v'' + 2w_{xx} + e^{-x/2}$. Letting $t \to \infty$, this becomes $0 = 2v'' + e^{-x/2}$.
- Plugging into (BC), we get $w_x(0,t) + v'(0) = 3$ and $w(4,t) + v(4) = -2$.
  Letting $t \to \infty$, these become $v'(0) = 3$ and $v(4) = -2$.
- Solving $0 = 2v'' + e^{-x/2}$, we find

  $$v(x) = \iint -\frac{1}{2} e^{-x/2} \mathrm{d}x \mathrm{d}x = \int e^{-x/2} \mathrm{d}x + C = -2e^{-x/2} + Cx + D.$$

  The boundary conditions $v'(0) = 3$ and $v(4) = -2$ imply $C = 2$ and $-2e^{-2} + 8 + D = -2$.
  and therefore the steady-state solution $v(x) = -2e^{-x/2} + 2x - 10 + 2e^{-2}$.

On the other hand, the transient solution $w(x,t)$ is characterized as the unique solution to:

$$\begin{array}{l} w_t = 2w_{xx} \\ w_x(0,t) = 0, \quad w(4,t) = 0 \\ w(x,0) = f(x) - v(x) \end{array}$$

Note. We know how to solve this homogeneous heat equation using separation of variables.