Though it requires some effort, we already know how to solve $p(D)y = F(t)$ for periodic forces $F(t)$, once we have a Fourier series for $F(t)$. The same approach works for linear equations of higher order, or even systems of equations.

Example 125. Find a particular solution of $2y'' + 32y = F(t)$, with $F(t) = \begin{cases} 10 & \text{if } t \in (0,1) \\ -10 & \text{if } t \in (1,2) \end{cases}$, extended 2-periodically.

Solution.

- From earlier, we already know $F(t) = 10\sum_{n \text{ odd}} \frac{4}{\pi n}\sin(\pi n t)$.
- We next solve the equation $2y'' + 32y = \sin(\pi n t)$ for $n = 1, 3, 5, \ldots$. First, we note that the external frequency is πn, which is never equal to the natural frequency $\omega_0 = 4$. Hence, there exists a particular solution of the form $y_p(t) = A\cos(\pi n t) + B\sin(\pi n t)$. To determine the coefficients A, B, we plug into the DE. Noting that $y_p'' = -\pi^2 n^2 y_p$ (why?!), we get

$$2y_p'' + 32y_p = (32 - 2\pi^2 n^2)(A\cos(\pi n t) + B\sin(\pi n t)) \overset{!}{=} \sin(\pi n t).$$

  We conclude $A = 0$ and $B = \frac{1}{32 - 2\pi^2 n^2}$, so that $y_p(t) = \frac{\sin(\pi n t)}{32 - 2\pi^2 n^2}$.
- We combine the particular solutions found in the previous step, to see that

$$2y'' + 32y = 10\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n}\sin(\pi n t) \quad \text{is solved by} \quad y_p = 10\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n}\frac{\sin(\pi n t)}{32 - 2\pi^2 n^2}.$$

Example 126. Find a particular solution of $2y'' + 32y = F(t)$, with $F(t)$ the 2π-periodic function such that $F(t) = 10t$ for $t \in (-\pi, \pi)$.

Solution.

- The Fourier series of $F(t)$ is $F(t) = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{20}{n}\sin(nt)$. [Exercise!]
- We next solve the equation $2y'' + 32y = \sin(nt)$ for $n = 1, 2, 3, \ldots$. Note, however, that **resonance** occurs for $n = 4$, so we need to treat that case separately. If $n \neq 4$ then we find, as in the previous example, that $y_p(t) = \frac{\sin(nt)}{32 - 2n^2}$. [Note how this fails for $n = 4$!]

  For $2y'' + 32y = \sin(4t)$, we begin with $y_p = At\cos(4t) + Bt\sin(4t)$. Then $y_p' = (A + 4Bt)\cos(4t) + (B - 4At)\sin(4t)$, and $y_p'' = (8B - 16At)\cos(4t) + (-8A - 16Bt)\sin(4t)$. Plugging into the DE, we get $2y_p'' + 32y_p = 16B\cos(4t) - 16A\sin(4t) \overset{!}{=} \sin(4t)$, and thus $B = 0$, $A = -\frac{1}{16}$. So, $y_p = -\frac{1}{16}t\cos(4t)$.
- We combine the particular solutions to get that our DE

$$2y'' + 32y = -5\sin(4t) + \sum_{\substack{n=1 \\ n \neq 4}}^{\infty} (-1)^{n+1}\frac{20}{n}\sin(nt)$$

  is solved by

$$y_p(t) = \frac{5}{16}t\cos(4t) + \sum_{\substack{n=1 \\ n \neq 4}}^{\infty} (-1)^{n+1}\frac{20}{n}\frac{\sin(nt)}{32 - 2n^2}.$$

  As in the previous example, this solution cannot really be simplified. Make some plots to appreciate the dominating character of the term resulting from resonance!

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