

Fourier series

The following amazing fact is saying that any 2π -periodic function can be written as a sum of cosines and sines.

Advertisement. In Linear Algebra II, we will see the following natural way to look at Fourier series: the functions $1, \cos(t), \sin(t), \cos(2t), \sin(2t), \dots$ are orthogonal to each other (for that to make sense, we need to think of functions as vectors and introduce a natural inner product). In fact, they form an orthogonal basis for the space of piecewise smooth functions. In that setting, the formulas for the coefficients a_n and b_n are nothing but the usual projection formulas for orthogonal projection onto a single vector.

Theorem 110. Every* 2π -periodic function f can be written as a **Fourier series**

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nt) + b_n \sin(nt)).$$

Technical detail*: f needs to be, e.g., piecewise smooth.

Also, if t is a discontinuity of f , then the Fourier series converges to the average $\frac{f(t^-) + f(t^+)}{2}$.

The **Fourier coefficients** a_n, b_n are unique and can be computed as

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) dt, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) dt.$$

Comment. Another common way to write Fourier series is $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$.

These two ways are equivalent; we can convert between them using Euler's identity $e^{int} = \cos(nt) + i \sin(nt)$.

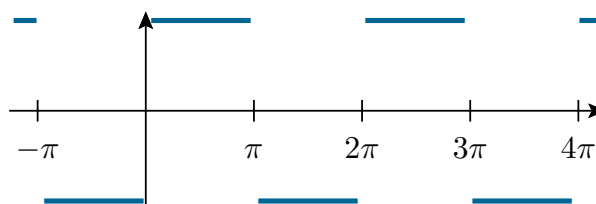
Definition 111. Let $L > 0$. $f(t)$ is **L -periodic** if $f(t+L) = f(t)$ for all t . The smallest such L is called the **(fundamental) period** of f .

Example 112. The fundamental period of $\cos(nt)$ is $2\pi/n$.

Example 113. The trigonometric functions $\cos(nt)$ and $\sin(nt)$ are 2π -periodic for every integer n . And so are their linear combinations. (Thus, 2π -periodic functions form a vector space!)

Example 114. Find the Fourier series of the 2π -periodic function $f(t)$ defined by

$$f(t) = \begin{cases} -1, & \text{for } t \in (-\pi, 0), \\ +1, & \text{for } t \in (0, \pi), \\ 0, & \text{for } t = -\pi, 0, \pi. \end{cases}$$



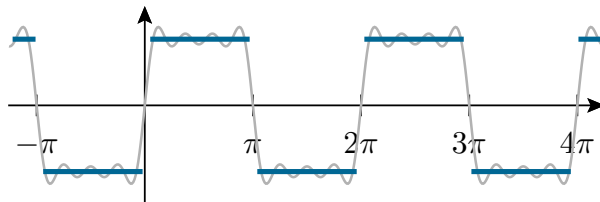
Solution. We compute $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) dt = 0$, as well as

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) dt = \frac{1}{\pi} \left[- \int_{-\pi}^0 \cos(nt) dt + \int_0^{\pi} \cos(nt) dt \right] = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) dt = \frac{1}{\pi} \left[- \int_{-\pi}^0 \sin(nt) dt + \int_0^{\pi} \sin(nt) dt \right] = \frac{2}{\pi n} [1 - \cos(n\pi)]$$

$$= \frac{2}{\pi n} [1 - (-1)^n] = \begin{cases} \frac{4}{\pi n} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}.$$

In conclusion, $f(t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \sin(nt) = \frac{4}{\pi} \left(\sin(t) + \frac{1}{3} \sin(3t) + \frac{1}{5} \sin(5t) + \dots \right)$.



Observation. The coefficients a_n are zero for all n if and only if $f(t)$ is odd.

Comment. The value of $f(t)$ for $t = -\pi, 0, \pi$ is irrelevant to the computation of the Fourier series. They are chosen so that $f(t)$ is equal to the Fourier series for all t (recall that, at a jump discontinuity t , the Fourier series converges to the average $\frac{f(t^-) + f(t^+)}{2}$).

Comment. Plot the (sum of the) first few terms of the Fourier series. What do you observe? The “overshooting” is known as the **Gibbs phenomenon**: https://en.wikipedia.org/wiki/Gibbs_phenomenon

Comment. Set $t = \frac{\pi}{2}$ in the Fourier series we just computed, to get Leibniz' series $\pi = 4 \left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \right]$. For such an alternating series, the error made by stopping at the term $1/n$ is on the order of $1/n$. To compute the 768 digits of π to get to the Feynman point (3.14159265...721134999999...), we would (roughly) need $1/n < 10^{-768}$, or $n > 10^{768}$. That's a lot of terms! (Roger Penrose, for instance, estimates that there are about 10^{80} atoms in the observable universe.)

Remark. Convergence of such series is not obvious! Recall, for instance, that the (odd part of) the harmonic series $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots$ diverges.

Fourier series with general period

There is nothing special about 2π -periodic functions considered before (except that $\cos(t)$ and $\sin(t)$ have fundamental period 2π). Note that $\cos(\pi t/L)$ and $\sin(\pi t/L)$ have period $2L$.

Theorem 115. Every $2L$ -periodic function f can be written as a **Fourier series**

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right) \right).$$

Technical detail*: f needs to be, e.g., piecewise smooth.

Also, if t is a discontinuity, then the Fourier series converges to the average $\frac{f(t^-) + f(t^+)}{2}$.

The **Fourier coefficients** a_n, b_n are unique and can be computed as

$$a_n = \frac{1}{L} \int_{-L}^L f(t) \cos\left(\frac{n\pi t}{L}\right) dt, \quad b_n = \frac{1}{L} \int_{-L}^L f(t) \sin\left(\frac{n\pi t}{L}\right) dt.$$

Important. This follows from Theorem 110 because, if $f(t)$ has period $2L$, then $\tilde{f}(x) := f\left(\frac{L}{\pi}t\right)$ has period 2π .

Example 116. Find the Fourier series of the 2-periodic function $g(t) = \begin{cases} -1 & \text{for } t \in (-1, 0) \\ +1 & \text{for } t \in (0, 1) \\ 0 & \text{for } t = -1, 0, 1 \end{cases}$.

Solution. Instead of computing from scratch, we can use the fact that $g(t) = f(\pi t)$, with f as in the previous example, to get $g(t) = f(\pi t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \sin(n\pi t)$.

Theorem 117. If $f(t)$ is **continuous** and $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right) \right)$, then* $f'(t) = \sum_{n=1}^{\infty} \left(\frac{n\pi}{L} b_n \cos\left(\frac{n\pi t}{L}\right) - \frac{n\pi}{L} a_n \sin\left(\frac{n\pi t}{L}\right) \right)$ (i.e., we can differentiate termwise).

Technical detail*: f' needs to be, e.g., piecewise smooth (so that it has a Fourier series itself).

Example 118. Let $h(t)$ be the 2-periodic function with $h(t) = |t|$ for $t \in [-1, 1]$. Compute the Fourier series of $h(t)$.

Solution. We could just use the integral formulas to compute a_n and b_n . Since $h(t)$ is even (plot it!), we will find that $b_n = 0$. Computing a_n is left as an exercise.

Solution. Note that $h(t) = \begin{cases} -t & \text{for } t \in (-1, 0) \\ +t & \text{for } t \in (0, 1) \end{cases}$ is continuous and $h'(t) = g(t)$, with $g(t)$ as in Example 116. Hence, we can apply Theorem 117 to conclude

$$h'(t) = g(t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \sin(n\pi t) \implies h(t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \left(-\frac{1}{\pi n} \right) \cos(n\pi t) + C,$$

where $C = \frac{a_0}{2} = \frac{1}{2} \int_{-1}^1 h(t) dt = \frac{1}{2}$ is the constant of integration. Thus, $h(t) = \frac{1}{2} - \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi^2 n^2} \cos(n\pi t)$.

Remark. Note that $t = 0$ in the last Fourier series, gives us $\frac{\pi^2}{8} = \frac{1}{1} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$. As an exercise, you can try to find from here the fact that $\sum_{n \geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}$. Similarly, we can use Fourier series to find that $\sum_{n \geq 1} \frac{1}{n^4} = \frac{\pi^4}{90}$.

Just for fun. These are the values $\zeta(2)$ and $\zeta(4)$ of the Riemann zeta function $\zeta(s)$. No such evaluations are known for $\zeta(3), \zeta(5), \dots$ and we don't even know (for sure) whether these are rational numbers. Nobody believes these to be rational numbers, but it was only in 1978 that Apéry proved that $\zeta(3)$ is not a rational number.

Example 119. (caution!) The function $g(t)$, from Example 116, is not continuous. For all values, except the discontinuities, we have $g'(t) = 0$. On the other hand, differentiating the Fourier series termwise, results in $4 \sum_{n \text{ odd}} \cos(n\pi t)$, which diverges for most values of t (that's easy to check for $t = 0$). This illustrates that we cannot apply Theorem 117 because of the missing continuity.

[The issues we are facing here can be fixed by generalizing the notion of function to distributions. (Maybe you have heard of the Dirac delta "function".)]