Example 99. (caution!) Consider the linear DE $x^2 y' = y - x$. Does it have a convergent power series solution at $x = 0$?

Important note. The DE $x^2 y' = y - x$ has the singular point $x = 0$. Hence, Theorem 92 does not apply.

Advanced. Moreover, in contrast to the previous example, $x = 0$ is not a **regular singular point**. Indeed, as we see below, there is no power series solution of the DE at all.

Solution. Let us look for a power series solution $y(x) = \sum_{n=0}^{\infty} a_n x^n$.

$x^2 y'(x) = x^2 \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n+1} = \sum_{n=2}^{\infty} (n-1) a_{n-1} x^n$

Hence, $x^2 y' = y - x$ becomes $\sum_{n=2}^{\infty} (n-1) a_{n-1} x^n = \sum_{n=0}^{\infty} a_n x^n - x$. We compare coefficients of x^n:

- $n = 0$: $a_0 = 0$.
- $n = 1$: $0 = a_1 - 1$, so that $a_1 = 1$.
- $n \geqslant 2$: $(n-1) a_{n-1} = a_n$, from which it follows that $a_n = (n-1) a_{n-1} = (n-1)(n-2) a_{n-2} = \cdots = (n-1)! a_1 = (n-1)!$.

Hence the DE has the "formal" power series solution $y(x) = \sum_{n=1}^{\infty} (n-1)! x^n$.

However, that series is divergent for all $x \neq 0$; that is, the radius of convergence is 0.

Inverses of power series

Example 100. (extra) For each of the following compute the first few terms of the power series.

(a) $(a_0 + a_1 x + a_2 x^2 + \ldots)(b_0 + b_1 x + b_2 x^2 \ldots)$

(b) $\dfrac{1}{a_0 + a_1 x + a_2 x^2 + \ldots}$

(c) $\dfrac{1}{1 + x + \frac{1}{2} x^2 + \frac{1}{6} x^3 + \ldots}$

Solution.

(a) $a_0 b_0 + (a_0 b_1 + a_1 b_0) x + (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2 + O(x^3)$

(b) The answer is $b_0 + b_1 x + \ldots$ with the property that $(a_0 + a_1 x + a_2 x^2 + \ldots)(b_0 + b_1 x + b_2 x^2 \ldots) = 1$.
By the first part, and comparing coefficients, $a_0 b_0 = 1$, $a_0 b_1 + a_1 b_0 = 0$, $a_0 b_2 + a_1 b_1 + a_2 b_0 = 0$, ...
Hence, $b_0 = \frac{1}{a_0}$, $b_1 = -\frac{1}{a_0}(a_1 b_0) = -\frac{a_1}{a_0^2}$, $b_2 = -\frac{1}{a_0}(a_1 b_1 + a_2 b_0) = \frac{a_1^2}{a_0^3} - \frac{a_2}{a_0^2}$.

(c) $\dfrac{1}{1 + x + \frac{1}{2} x^2 + \frac{1}{6} x^3 + \ldots} = 1 - x + \dfrac{1}{2} x^2 - \dfrac{1}{6} x^3 + \ldots$

Comment. This reflects $\frac{1}{e^x} = e^{-x}$.

Armin Straub
straub@southalabama.edu

Likewise, we could compute the first few terms of the power series of, say, $\frac{1}{1-x-x^2}$.

However, it turns out that we can describe all terms in that power series:

Example 101. Derive a recursive description of the power series for $y(x) = \frac{1}{1-x-x^2}$.

Solution. Write $y(x) = \sum_{n=0}^{\infty} a_n x^n$. Then

$$\begin{aligned} 1 = (1-x-x^2)\sum_{n=0}^{\infty} a_n x^n &= \sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+2} \\ &= \sum_{n=0}^{\infty} a_n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n - \sum_{n=2}^{\infty} a_{n-2} x^n. \end{aligned}$$

We compare coefficients of x^n:

- $n=0$: $1 = a_0$.
- $n=1$: $0 = a_1 - a_0$, so that $a_1 = a_0 = 1$.
- $n \geqslant 2$: $0 = a_n - a_{n-1} - a_{n-2}$ or, equivalently, $a_n = a_{n-1} + a_{n-2}$.

This is the recursive description of the Fibonacci numbers F_n! In particular $a_n = F_n$.

The first few terms. $\frac{1}{1-x-x^2} = 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + 13x^6 + \ldots$

Comment. The function $y(x)$ is said to be a **generating function** for the Fibonacci numbers.

Challenge. Can you rederive Binet's formula from partial fractions and the geometric series?