Review. Theorem 92: If x_0 is an ordinary point of a linear IVP, then it is guaranteed to have a power series solution $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$.

Moreover, its radius of convergence is at least the distance between x_0 and the closest singular point.

Example 96. Find a minimum value for the radius of convergence of a power series solution to $(x^2 + 4)y'' - 3xy' + \frac{1}{x+1} y = 0$ at $x = 2$.

Solution. The singular points are $x = \pm 2i, -1$. Hence, $x = 2$ is an ordinary point of the DE and the distance to the nearest singular point is $|2 - 2i| = \sqrt{2^2 + 2^2} = \sqrt{8}$ (the distances are $|2 - (-1)| = 3$, $|2 - 2i| = |2 - (-2i)| = \sqrt{8}$).
By Theorem 92, the DE has power series solutions about $x = 2$ with radius of convergence at least $\sqrt{8}$.

Example 97. (caution!) Theorem 92 only holds for linear DEs! For nonlinear DEs, it is very hard to predict whether there is a power series solution and what its radius of convergence is.
Consider, for instance, the nonlinear DE $y' + 2xy^2 = 0$.

Its coefficients have no singularities.

A solution to this DE is $y(x) = \frac{1}{1 + x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$ (check that!), which has radius of convergence 1.

On the other hand. $y(x)$ also solves the linear DE $(1 + x^2)y' + 2xy = 0$. Note how the DE has singular points for $x = \pm i$. This allows us to predict that $y(x)$ must have a power series with radius of convergence at least 1.

Example 98. (Bessel functions) Consider the DE $x^2 y'' + xy' + x^2 y = 0$. Derive a recursive description of a power series solutions $y(x)$ at $x = 0$.

Caution! Note that $x = 0$ is a singular point (the only) of the DE. Theorem 92 therefore does not guarantee a basis of power series solutions. [However, $x = 0$ is what is called a **regular singular point**; for these, we are guaranteed one power series solution, as well as additional solutions expressed using logarithms and power series.]

Comment. We could divide the DE by x (but that wouldn't really change the computations below). The reason for not dividing that x is that this DE is the special case $\alpha = 0$ of the **Bessel equation** $x^2 y'' + xy' + (x^2 - \alpha^2) y = 0$ (for which no such dividing is possible).

Solution. (plug in power series) Let us spell out power series for $x^2 y, xy', x^2 y''$ starting with $y(x) = \sum_{n=0}^{\infty} a_n x^n$:

$$x^2 y(x) = \sum_{n=0}^{\infty} a_n x^{n+2} = \sum_{n=2}^{\infty} a_{n-2} x^n$$

$$xy'(x) = \sum_{n=1}^{\infty} n a_n x^n \qquad \left(\text{because } y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}\right)$$

$$x^2 y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^n \qquad \left(\text{because } y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right)$$

Hence, the DE becomes $\sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0$. We compare coefficients of x^n:

- $n = 1$: $a_1 = 0$
- $n \geqslant 2$: $n(n-1) a_n + n a_n + a_{n-2} = 0$, which simplifies to $n^2 a_n = -a_{n-2}$.
  It follows that $a_{2n} = \frac{(-1)^n}{4^n n!^2} a_0$ and $a_{2n+1} = 0$.

Observation. The fact that we found $a_1 = 0$ reflects the fact that we cannot represent the general solution through power series alone.

Comment. If $a_0 = 1$, the function we found is a **Bessel function** and denoted as $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!^2} \left(\frac{x}{2}\right)^{2n}$.

The more general Bessel functions $J_\alpha(x)$ are solutions to the DE $x^2 y'' + xy' + (x^2 - \alpha^2) y = 0$.

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