

Once we have a power series solution  $y(x)$ , a natural question is: for which  $x$  does the series converge?

**Recall.** A power series  $y(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$  has a **radius of convergence**  $R$ .

The series converges for all  $x$  with  $|x-x_0| < R$  and it diverges for all  $x$  with  $|x-x_0| > R$ .

**Definition 89.** Consider the linear DE  $y^{(n)} + p_{n-1}(x)y^{(n-1)} + \dots + p_1(x)y' + p_0(x)y = f(x)$ .  $x_0$  is called an **ordinary point** if the coefficients  $p_j(x)$ , as well as  $f(x)$ , are analytic at  $x = x_0$ . Otherwise,  $x_0$  is called a **singular point**.

**Example 90.** Determine the singular points of  $(x+2)y'' - x^2y + 3y = 0$ .

**Solution.** Rewriting the DE as  $y'' - \frac{x^2}{x+2}y + \frac{3}{x+2}y = 0$ , we see that the only singular point is  $x = -2$ .

**Example 91.** Determine the singular points of  $(x^2+1)y''' = \frac{y}{x-5}$ .

**Solution.** Rewriting the DE as  $y''' - \frac{1}{(x-5)(x^2+1)}y = 0$ , we see that the singular points are  $x = \pm i, 5$ .

**Theorem 92.** Consider the linear DE  $y^{(n)} + p_{n-1}(x)y^{(n-1)} + \dots + p_1(x)y' + p_0(x)y = f(x)$ . Suppose that  $x_0$  is an ordinary point and that  $R$  is the distance to the closest singular point. Then any IVP specifying  $y(x_0), y'(x_0), \dots, y^{(n-1)}(x_0)$  has a power series solution  $y(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$  and that series has radius of convergence at least  $R$ .

**In particular.** The DE has a general solution consisting of  $n$  solutions  $y(x)$  that are analytic at  $x = x_0$ .

**Comment.** Most textbooks only discuss the case of 2nd order DEs. For a discussion of the higher order case (in terms of first order systems!) see, for instance, Chapter 4.5 in *Ordinary Differential Equations* by N. Lebovitz.

The book is freely available at: <http://people.cs.uchicago.edu/~lebovitz/odes.html>

**Example 93.** Find a minimum value for the radius of convergence of a power series solution to  $(x+2)y'' - x^2y' + 3y = 0$  at  $x = 3$ .

**Solution.** As before, rewriting the DE as  $y'' - \frac{x^2}{x+2}y' + \frac{3}{x+2}y = 0$ , we see that the only singular point is  $x = -2$ .

Note that  $x = 3$  is an ordinary point of the DE and that the distance to the singular point is  $|3 - (-2)| = 5$ .

Hence, the DE has power series solutions about  $x = 3$  with radius of convergence at least 5.

**Example 94.** Find a minimum value for the radius of convergence of a power series solution to  $(x^2+1)y''' = \frac{y}{x-5}$  at  $x = 2$ .

**Solution.** As before, rewriting the DE as  $y''' - \frac{1}{(x-5)(x^2+1)}y = 0$ , we see that the singular points are  $x = \pm i, 5$ .

Note that  $x = 2$  is an ordinary point of the DE and that the distance to the nearest singular point is  $|2 - i| = \sqrt{5}$  (the distances are  $|2 - 5| = 3, |2 - i| = |2 - (-i)| = \sqrt{2^2 + 1^2} = \sqrt{5}$ ).

Hence, the DE has power series solutions about  $x = 2$  with radius of convergence at least  $\sqrt{5}$ .

**Example 95. (Airy equation, once more)** Let  $y(x)$  be the solution to the IVP  $y'' = xy, y(0) = a, y'(0) = b$ . Earlier, we determined the power series of  $y(x)$ . What is its radius of convergence?

**Solution.**  $y'' = xy$  has no singular points. Hence, the radius of convergence is  $\infty$ . (In other words, the power series converges for all  $x$ .)