Power series solutions to linear DEs

Note how in the last two examples the "plug in power series" approach was complicated by the fact that the DE was not linear (we had to expand y^2 as well as $\cos(x+y)$, respectively).

For linear DEs, this complication does not arise and we can readily determine the complete power series expansion of analytic solutions (with a recursive description of the coefficients).

Example 88. (Airy equation, cont'd) Let $y(x)$ be the unique solution to the IVP $y'' = xy$, $y(0) = a$, $y'(0) = b$. Determine the power series of $y(x)$.

Solution. (plug in power series) Let us spell out the power series for y, y', y'' and xy:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$xy(x) = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1} x^n$$

Hence, $y'' = xy$ becomes $\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n = \sum_{n=1}^{\infty} a_{n-1} x^n$. We compare coefficients of x^n:

- $n = 0$: $2 \cdot 1 a_2 = 0$, so that $a_2 = 0$.
- $n \geqslant 1$: $(n+2)(n+1) a_{n+2} = a_{n-1}$
  Replacing n by $n-2$, this is equivalent to $n(n-1) a_n = a_{n-3}$ for $n \geqslant 3$.

In conclusion, $y(x) = \sum_{n=0}^{\infty} a_n x^n$ with $a_0 = a$, $a_1 = b$, $a_2 = 0$ as well as, for $n \geqslant 3$, $a_n = \frac{1}{n(n-1)} a_{n-3}$.

First few terms. In particular, $y = a\left(1 + \frac{x^3}{2 \cdot 3} + \frac{x^6}{(2 \cdot 3)(5 \cdot 6)} + \ldots\right) + b\left(x + \frac{x^4}{3 \cdot 4} + \frac{x^7}{(3 \cdot 4)(6 \cdot 7)} + \ldots\right)$.

Advanced. The solution with $y(0) = \frac{1}{3^{2/3}\Gamma(2/3)}$ and $y'(0) = -\frac{1}{3^{1/3}\Gamma(1/3)}$ is known as the **Airy function** $\mathrm{Ai}(x)$.
[A more natural property of $\mathrm{Ai}(x)$ is that it satisfies $y(x) \to 0$ as $x \to \infty$.]

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