Some special functions and the power series method

Review: power series

Definition 81. A function $y(x)$ is analytic around $x = x_0$ if it has a power series

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

Note. In the next theorem, we will see that this power series is the Taylor series of $y(x)$ around $x = x_0$.

Power series are very pleasant to work with because they behave just like polynomials. For instance, we can differentiate and integrate them:

- If $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$, then $y'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}$ (another power series!).

  Note that $y'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} (x - x_0)^n$. Likewise, for higher derivatives.

- $\int y(x) \mathrm{d}x = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x - x_0)^{n+1} + C$

Theorem 82. If $y(x)$ is analytic around $x = x_0$, then $y(x)$ is infinitely differentiable and

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad \text{with} \quad a_n = \frac{y^{(n)}(x_0)}{n!}.$$

Caution. Analyticity is needed in this theorem; being infinitely differentiable is not enough. For instance, $y(x) = e^{-1/x^2}$ is infinitely differentiable around $x = 0$ (and everywhere else). However, $y^{(n)}(0) = 0$ for all n.

We have already seen the following example.

Example 83. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{1}{2} x^2 + \frac{1}{3!} x^3 + \ldots$

Once again, notice how the power series clearly has the property that $y' = y$.

It follows from here that, for instance, $e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = 1 + 2x + 2x^2 + \frac{4}{3} x^3 + \ldots$

Armin Straub
straub@southalabama.edu

Example 84. Determine a power series for $\cos(x)$.

Solution. (via DE) $\cos(x)$ is the unique solution to the IVP $y'' = -y$, $y(0) = 1$, $y'(0) = 0$.

It follows that $\cos(x) = \sum_{n=0}^{\infty} a_n x^n$ with $a_n = \frac{y^{(n)}(0)}{n!}$. The DE implies that $y^{(2n)}(x) = (-1)^n y(x)$ and $y^{(2n+1)} = (-1)^n y'(x)$ so that $y^{(2n)}(0) = (-1)^n$ and $y^{(2n+1)}(0) = 0$. Consequently, $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$.

Solution. (via Euler's formula) Recall that $e^{ix} = \cos(x) + i\sin(x)$. Since

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{m=0}^{\infty} \frac{(ix)^{2m}}{(2m)!} + \sum_{m=0}^{\infty} \frac{(ix)^{2m+1}}{(2m+1)!} = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} + i\sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!},$$

we conclude that $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$ and $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$.