Modeling

Example 78. Consider two brine tanks. Tank T_1 contains 24gal water containing 3lb salt, and tank T_2 contains 9gal pure water.

- T_1 is being filled with 54gal/min water containing 0.5lb/gal salt.
- 72gal/min well-mixed solution flows out of T_1 into T_2.
- 18gal/min well-mixed solution flows out of T_2 into T_1.
- Finally, 54gal/min well-mixed solution is leaving T_2.

How much salt is in the tanks after t minutes?

Solution. Note that the amount of water in each tank is constant because the flows balance each other. Let $y_i(t)$ denote the amount of salt (in lb) in tank T_i after time t (in min). In time interval $[t, t+\Delta t]$:

$\Delta y_1 \approx 54 \cdot \frac{1}{2} \cdot \Delta t - 72 \cdot \frac{y_1}{24} \cdot \Delta t + 18 \cdot \frac{y_2}{9} \cdot \Delta t$, so $y_1' = 27 - 3y_1 + 2y_2$. Also, $y_1(0) = 3$.

$\Delta y_2 \approx 72 \cdot \frac{y_1}{24} \cdot \Delta t - 72 \cdot \frac{y_2}{9} \cdot \Delta t$, so $y_2' = 3y_1 - 8y_2$. Also, $y_2(0) = 0$.

Using matrix notation and writing $\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, this is $\boldsymbol{y}' = \begin{bmatrix} -3 & 2 \\ 3 & -8 \end{bmatrix} \boldsymbol{y} + \begin{bmatrix} 27 \\ 0 \end{bmatrix}$, $\boldsymbol{y}(0) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$.

This is an IVP that we can solve (with some work)! Do it! Skipping most work, we find:

- $e^{At} = \frac{1}{7} \begin{bmatrix} e^{-9t} + 6e^{-2t} & -2e^{-9t} + 2e^{-2t} \\ -3e^{-9t} + 3e^{-2t} & 6e^{-9t} + 1e^{-2t} \end{bmatrix}$
- $\boldsymbol{y} = e^{At} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{At} \int_0^t e^{-As} \begin{bmatrix} 27 \\ 0 \end{bmatrix} \mathrm{d}s = \frac{1}{7} \begin{bmatrix} e^{-9t} + 6e^{-2t} \\ -3e^{-9t} + 3e^{-2t} \end{bmatrix} + \frac{3}{14} e^{At} \begin{bmatrix} 2e^{9t} + 54e^{2t} - 56 \\ -6e^{9t} + 27e^{2t} - 21 \end{bmatrix} = \begin{bmatrix} 12 - 9e^{-2t} \\ 4.5 - 4.5e^{-2t} \end{bmatrix}$

Note. We could have found a particular solution with less calculations by observing (looking at "old" and "new" roots) that there must be a solution of the form $\boldsymbol{y}_p(t) = \boldsymbol{a}$. We can then find $\boldsymbol{a}$ by plugging into the differential equation. However, noticing that, for a constant solution, each tank has to have a constant concentration of 0.5lb/gal of salt, we find $\boldsymbol{y}_p(t) = \begin{bmatrix} 12 \\ 4.5 \end{bmatrix}$.

Two more applications of systems of DEs

Example 79. (epidemiology) Let us indicate the popular SIR model for short outbreaks of diseases among a population of constant size N.

In a SIR model, the population is compartmentalized into $S(t)$ susceptible, $I(t)$ infected and $R(t)$ recovered (or resistant) individuals ($N = S(t) + I(t) + R(t)$). In the Kermack-McKendrick model, the outbreak of a disease is modeled by

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \gamma I, \quad \frac{\mathrm{d}S}{\mathrm{d}t} = -\beta S I, \quad \frac{\mathrm{d}I}{\mathrm{d}t} = \beta S I - \gamma I,$$

with γ modeling the recovery rate and β the infection rate. Note that this is a non-linear system of differential equations. For more details and many variations used in epidemiology, see:

`https://en.wikipedia.org/wiki/Compartmental_models_in_epidemiology`

Comment. The following variation

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \gamma I R, \quad \frac{\mathrm{d}S}{\mathrm{d}t} = -\beta S I, \quad \frac{\mathrm{d}I}{\mathrm{d}t} = \beta S I - \gamma I R,$$

which assumes "infectious recovery", was used in 2014 to predict that facebook might lose 80% of its users by 2017. It's that claim, not mathematics (or even the modeling), which attracted a lot of media attention.

`http://blogs.wsj.com/digits/2014/01/22/controversial-paper-predicts-facebook-decline/`

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Example 80. (military strategy) Lanchester's equations model two opposing forces during "aimed fire" battle.

Let $x(t)$ and $y(t)$ describe the number of troops on each side. Then Lanchester (during World War I) assumed that the rates $-x'(t)$ and $-y'(t)$, at which soldiers are put out of action, are proportional to the number of opposing forces. That is:

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -\alpha y(t) \\ -\beta x(t) \end{bmatrix}, \quad \text{or, in matrix form: } \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -\alpha \\ -\beta & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

The proportionality constants $\alpha, \beta > 0$ indicate the strength of the forces ("fighting effectiveness coefficients").

These are simple linear DEs with constant coefficients, which we have learned how to solve.

For more details, see: `https://en.wikipedia.org/wiki/Lanchester%27s_laws`

Comment. The "aimed fire" means that all combatants are engaged, as is common in modern combat with long-range weapons. This is rather different than ancient combat where soldiers were engaging one opponent at a time.

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