Systems of linear DEs: the inhomogeneous case

Recall that any linear DE can be transformed into a first-order system. Hence, any linear DE (or any system of linear DEs) can written as

$$\boldsymbol{y}' = A(x)\,\boldsymbol{y} + \boldsymbol{f}(x).$$

Note. In general, A depends on x. In other words, the DE is allowed to have nonconstant coefficients.

Review. We showed in Theorem 15 that $y' = a(x)y + f(x)$ has the particular solution

$$y_p(x) = y_h(x)\int \frac{f(x)}{y_h(x)}\mathrm{d}x,$$

where $y_h(x) = e^{\int a(x)\mathrm{d}x}$ is any solution to the homogeneous equation $y' = a(x)y$.

Amazingly (or, maybe, by now, not surprisingly), the same arguments with the same result apply to systems of linear equations:

Theorem 75. (variation of constants) $\boldsymbol{y}' = A(x)\,\boldsymbol{y} + \boldsymbol{f}(x)$ has the particular solution

$$\boldsymbol{y}_p(x) = \Phi(x)\int \Phi(x)^{-1}\boldsymbol{f}(x)\mathrm{d}x,$$

where $\Phi(x)$ is any fundamental matrix solution to $\boldsymbol{y}' = A(x)\,\boldsymbol{y}$.

Proof. We can find this formula in the same manner as we did in Theorem 15:

Since the general solution of the homogeneous equation $\boldsymbol{y}' = A(x)\,\boldsymbol{y}$ is $\boldsymbol{y}_h = \Phi(x)\boldsymbol{c}$, we are going to vary the constant $\boldsymbol{c}$ and look for a particular solution of the form $\boldsymbol{y}_p = \Phi(x)\boldsymbol{c}(x)$. Plugging into the DE, we get:

$\boldsymbol{y}_p' = \Phi'\boldsymbol{c} + \Phi\boldsymbol{c}' = A\Phi\boldsymbol{c} + \Phi\boldsymbol{c}' \overset{!}{=} A\boldsymbol{y}_p + \boldsymbol{f} = A\Phi\boldsymbol{c} + \boldsymbol{f}$

For the first equality, we used the matrix version of the usual product rule (which holds since differentiation is defined entry-wise). For the second equality, we used $\Phi' = A\Phi$.

Hence, $\boldsymbol{y}_p = \Phi(x)\boldsymbol{c}(x)$ is a particular solution if and only if $\Phi\boldsymbol{c}' = \boldsymbol{f}$.

The latter condition means $\boldsymbol{c}' = \Phi^{-1}\boldsymbol{f}$ so that $\boldsymbol{c} = \int \Phi(x)^{-1}\boldsymbol{f}(x)\mathrm{d}x$, which gives the claimed formula for $\boldsymbol{y}_p$. □

Example 76. Find a particular solution to $\boldsymbol{y}' = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}\boldsymbol{y} + \begin{bmatrix} 0 \\ -2e^{3x} \end{bmatrix}$.

Solution. First, we determine (do it!) a fundamental matrix solution for $\boldsymbol{y}' = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}\boldsymbol{y}$: $\Phi(x) = \begin{bmatrix} e^{-x} & 3e^{4x} \\ -e^{-x} & 2e^{4x} \end{bmatrix}$

Using $\det(\Phi(x)) = 5e^{3x}$, we find $\Phi(x)^{-1} = \frac{1}{5}\begin{bmatrix} 2e^{x} & -3e^{x} \\ e^{-4x} & e^{-4x} \end{bmatrix}$.

Hence, $\Phi(x)^{-1}\boldsymbol{f}(x) = \frac{2}{5}\begin{bmatrix} 3e^{4x} \\ -e^{-x} \end{bmatrix}$ and $\int \Phi(x)^{-1}\boldsymbol{f}(x)\mathrm{d}x = \frac{2}{5}\begin{bmatrix} 3/4e^{4x} \\ e^{-x} \end{bmatrix}$.

By variation of constants, $\boldsymbol{y}_p(x) = \Phi(x)\int \Phi(x)^{-1}\boldsymbol{f}(x)\mathrm{d}x = \begin{bmatrix} e^{-x} & 3e^{4x} \\ -e^{-x} & 2e^{4x} \end{bmatrix}\frac{2}{5}\begin{bmatrix} 3/4e^{4x} \\ e^{-x} \end{bmatrix} = \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix}e^{3x}$.

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In the special case that $\Phi(x) = e^{Ax}$, some things become easier. For instance, $\Phi(x)^{-1} = e^{-Ax}$. In that case, we can explicitely write down solutions to IVPs:

- $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{c}$ has (unique) solution $\boldsymbol{y}(x) = e^{Ax}\boldsymbol{c}$.
- $\boldsymbol{y}' = A\boldsymbol{y} + \boldsymbol{f}(x)$, $\boldsymbol{y}(0) = \boldsymbol{c}$ has (unique) solution $\boldsymbol{y}(x) = e^{Ax}\boldsymbol{c} + e^{Ax}\int_0^x e^{-At}\boldsymbol{f}(t)\mathrm{d}t$.

Example 77. Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$.

(a) Determine e^{Ax}.

(b) Solve $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(c) Solve $\boldsymbol{y}' = A\boldsymbol{y} + \begin{bmatrix} 0 \\ 2e^x \end{bmatrix}$, $\boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(d) Given e^{Ax} (but not A), how can we find A?

Solution.

(a) By proceeding as in Example 67 (do it!), we find $e^{Ax} = \begin{bmatrix} 2e^{2x} - e^{3x} & -2e^{2x} + 2e^{3x} \\ e^{2x} - e^{3x} & -e^{2x} + 2e^{3x} \end{bmatrix}$.

(b) $\boldsymbol{y}(x) = e^{Ax}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2e^{2x} + 3e^{3x} \\ -e^{2x} + 3e^{3x} \end{bmatrix}$

(c) $\boldsymbol{y}(x) = e^{Ax}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + e^{Ax}\int_0^x e^{-At}\boldsymbol{f}(t)\mathrm{d}t$. We compute:

$\int_0^x e^{-At}\boldsymbol{f}(t)\mathrm{d}t = \int_0^x \begin{bmatrix} 2e^{-2t} - e^{-3t} & -2e^{-2t} + 2e^{-3t} \\ e^{-2t} - e^{-3t} & -e^{-2t} + 2e^{-3t} \end{bmatrix}\begin{bmatrix} 0 \\ 2e^t \end{bmatrix}\mathrm{d}t = \int_0^x \begin{bmatrix} -4e^{-t} + 4e^{-2t} \\ -2e^{-t} + 4e^{-2t} \end{bmatrix}\mathrm{d}t = \begin{bmatrix} 4e^{-x} - 2e^{-2x} - 2 \\ 2e^{-x} - 2e^{-2x} \end{bmatrix}$

Hence, $e^{Ax}\int_0^x e^{-At}\boldsymbol{f}(t)\mathrm{d}t = \begin{bmatrix} 2e^{2x} - e^{3x} & -2e^{2x} + 2e^{3x} \\ e^{2x} - e^{3x} & -e^{2x} + 2e^{3x} \end{bmatrix}\begin{bmatrix} 4e^{-x} - 2e^{-2x} - 2 \\ 2e^{-x} - 2e^{-2x} \end{bmatrix} = \begin{bmatrix} 2e^x - 4e^{2x} + 2e^{3x} \\ -2e^{2x} + 2e^{3x} \end{bmatrix}$.

Finally, $\boldsymbol{y}(x) = \begin{bmatrix} -2e^{2x} + 3e^{3x} \\ -e^{2x} + 3e^{3x} \end{bmatrix} + \begin{bmatrix} 2e^x - 4e^{2x} + 2e^{3x} \\ -2e^{2x} + 2e^{3x} \end{bmatrix} = \begin{bmatrix} 2e^x - 6e^{2x} + 5e^{3x} \\ -3e^{2x} + 5e^{3x} \end{bmatrix}$.

Sage. Here is how we can let Sage do these computations for us:

```
>>> t = var('t')
>>> A = matrix([[1,2],[-1,4]])
>>> y = exp(A*x)*vector([1,2]) + exp(A*x)*integrate(exp(-A*t)*vector([0,2*e^t]), t,0,x)
>>> y.simplify_full()
```

$(5\ e^{(3\,x)} - 6\ e^{(2\,x)} + 2\ e^x,\ 5\ e^{(3\,x)} - 3\ e^{(2\,x)})$

(d) Like any fundamental matrix, $\Phi = e^{Ax}$ satisfies $\Phi' = A\Phi$, that is $\frac{\mathrm{d}}{\mathrm{d}x}e^{Ax} = Ae^{Ax}$.

Hence, $A = \left[\frac{\mathrm{d}}{\mathrm{d}x}e^{Ax}\right]_{x=0} = \left[\begin{bmatrix} 4e^{2x} - 3e^{3x} & -4e^{2x} + 6e^{3x} \\ 2e^{2x} - 3e^{3x} & -2e^{2x} + 6e^{3x} \end{bmatrix}\right]_{x=0} = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$.

Alternatively. We saw earlier that we can easily convert between e^{Ax} and A^n.

In the present case, we find $A^n = \begin{bmatrix} 2\cdot 2^n - 3^n & -2\cdot 2^n + 2\cdot 3^n \\ 2^n - 3^n & -2^n + 2\cdot 3^n \end{bmatrix}$. In particular, $A = A^1 = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$.