(exponential function) e^x is the unique solution to $y' = y$, $y(0) = 1$.
From here, it follows that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$.

The latter is the Taylor series for e^x at $x = 0$ that we have seen in Calculus II.
Important note. We can actually construct this infinite sum directly from $y' = y$ and $y(0) = 1$.
Indeed, observe how each term, when differentiated, produces the term before it. For instance, $\frac{\mathrm{d}}{\mathrm{d}x} \frac{x^3}{3!} = \frac{x^2}{2!}$.

Review. We defined the **matrix exponential** e^{Mx} as the unique matrix solution to the IVP

$$\boldsymbol{y}' = M\boldsymbol{y}, \quad \boldsymbol{y}(0) = I.$$

Below, we observe that we can also make sense of the matrix exponential e^{Mx} as a power series.

Theorem 71. Let M be $n \times n$. Then the **matrix exponential** satisfies

$$e^M = I + M + \frac{1}{2!} M^2 + \frac{1}{3!} M^3 + \ldots$$

Proof. Define $\Phi(x) = I + Mx + \frac{1}{2!} M^2 x^2 + \frac{1}{3!} M^3 x^3 + \ldots$

$$\begin{aligned}\Phi'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left[I + Mx + \frac{1}{2!} M^2 x^2 + \frac{1}{3!} M^3 x^3 + \ldots \right] \\ &= 0 + M + M^2 x + \frac{1}{2!} M^3 x^2 + \ldots = M\Phi(x).\end{aligned}$$

Clearly, $\Phi(0) = I$. Therefore, $\Phi(x)$ is the fundamental matrix solution to $\boldsymbol{y}' = M\boldsymbol{y}$, $\boldsymbol{y}(0) = I$.
But that's precisely how we defined e^{Mx} earlier. It follows that $\Phi(x) = e^{Mx}$. Now set $x = 1$. □

Example 72. If $A = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$, then $A^{100} = \begin{bmatrix} 2^{100} & 0 \\ 0 & 5^{100} \end{bmatrix}$.

Example 73. If $A = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$, then $e^A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} 2^2 & 0 \\ 0 & 5^2 \end{bmatrix} + \cdots = \begin{bmatrix} e^2 & 0 \\ 0 & e^5 \end{bmatrix}$.

Clearly, this works to obtain e^D for every diagonal matrix D.

In particular, for $Ax = \begin{bmatrix} 2x & 0 \\ 0 & 5x \end{bmatrix}$, $e^{Ax} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2x & 0 \\ 0 & 5x \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} (2x)^2 & 0 \\ 0 & (5x)^2 \end{bmatrix} + \cdots = \begin{bmatrix} e^{2x} & 0 \\ 0 & e^{5x} \end{bmatrix}$.

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Review. To construct a fundamental matrix solution $\Phi(x)$ to $\boldsymbol{y}' = M\boldsymbol{y}$, we compute eigenvectors:
Given a λ-eigenvector $\boldsymbol{v}$, we have the corresponding solution $\boldsymbol{y}(x) = \boldsymbol{v}e^{\lambda x}$.
If there are enough eigenvectors, we can collect these as columns to obtain $\Phi(x)$.
The next example illustrates how to proceed if there are not enough eigenvectors.

In that case, instead of looking only for solutions of the type $\boldsymbol{y}(x) = \boldsymbol{v}e^{\lambda x}$, we also need to look for solutions of the type $\boldsymbol{y}(x) = (\boldsymbol{v}x + \boldsymbol{w})e^{\lambda x}$. This can only happen if an eigenvalue is a repeated root of the characteristic polynomial.

Example 74. Let $M = \begin{bmatrix} 8 & 4 \\ -1 & 4 \end{bmatrix}$.

(a) Determine the general solution to $\boldsymbol{y}' = M\boldsymbol{y}$.

(b) Determine a fundamental matrix solution to $\boldsymbol{y}' = M\boldsymbol{y}$.

(c) Compute e^{Mx}.

(d) Solve the initial value problem $\boldsymbol{y}' = M\boldsymbol{y}$ with $\boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Solution.

(a) We determine the eigenvectors of M. The characteristic polynomial is:
$\det(M - \lambda I) = \det\left(\begin{bmatrix} 8-\lambda & 4 \\ -1 & 4-\lambda \end{bmatrix}\right) = (8-\lambda)(4-\lambda) + 4 = \lambda^2 - 12\lambda + 36 = (\lambda - 6)(\lambda - 6)$
Hence, the eigenvalues are $\lambda = 6, 6$ (meaning that 6 has multiplicity 2).

- To find eigenvectors $\boldsymbol{v}$ for $\lambda = 6$, we need to solve $\begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}\boldsymbol{v} = \boldsymbol{0}$.
  Hence, $\boldsymbol{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 6$. There is no independent second eigenvector.

- We therefore search for a solution of the form $\boldsymbol{y}(x) = (\boldsymbol{v}x + \boldsymbol{w})e^{\lambda x}$ with $\lambda = 6$.
  $\boldsymbol{y}'(x) = (\lambda\boldsymbol{v}x + \lambda\boldsymbol{w} + \boldsymbol{v})e^{\lambda x} \overset{!}{=} M\boldsymbol{y} = (M\boldsymbol{v}x + M\boldsymbol{w})e^{\lambda x}$
  Equating coefficients of x, we need $\lambda\boldsymbol{v} = M\boldsymbol{v}$ and $\lambda\boldsymbol{w} + \boldsymbol{v} = M\boldsymbol{w}$.
  Hence, $\boldsymbol{v}$ must be an eigenvector (which we already computed); we choose $\boldsymbol{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.
  [Note that any multiple of $\boldsymbol{y}(x)$ will be another solution, so it doesn't matter which multiple of $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ we choose.]
  $\lambda\boldsymbol{w} + \boldsymbol{v} = M\boldsymbol{w}$ or $(M - \lambda)\boldsymbol{w} = \boldsymbol{v}$ then becomes $\begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}\boldsymbol{w} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.
  One solution is $\boldsymbol{w} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. [We only need one.]

Hence, the general solution is $C_1\begin{bmatrix} -2 \\ 1 \end{bmatrix}e^{6x} + C_2\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}x + \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right)e^{6x}$.

(b) The corresponding fundamental matrix solution is $\Phi = \begin{bmatrix} -2e^{6x} & -(2x+1)e^{6x} \\ e^{6x} & x\,e^{6x} \end{bmatrix}$.

(c) Note that $\Phi(0) = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix}$, so that $\Phi(0)^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$. It follows that

$$e^{Mx} = \Phi(x)\Phi(0)^{-1} = \begin{bmatrix} -2e^{6x} & -(2x+1)e^{6x} \\ e^{6x} & x\,e^{6x} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} (2x+1)e^{6x} & 4x\,e^{6x} \\ -x\,e^{6x} & -(2x-1)\,e^{6x} \end{bmatrix}.$$

(d) The solution to the IVP is $\boldsymbol{y}(x) = e^{Mx}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} (2x+1)e^{6x} & 4x\,e^{6x} \\ -x\,e^{6x} & -(2x-1)\,e^{6x} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} (2x+1)e^{6x} \\ -x\,e^{6x} \end{bmatrix}$.