Sketch of Lecture 14

Fri, 2/10/2023

Systems of differential equations

Example 63. Write the (second-order) differential equation $y'' = 2y' + y$ as a system of (first-order) differential equations.

Solution. Write $y_1 = y$ and $y_2 = y'$. Then $y'' = 2y' + y$ becomes $y_2' = 2y_2 + y_1$.

Therefore, $y'' = 2y' + y$ translates into the first-order system $\begin{cases} y_1' = y_2 \\ y_2' = y_1 + 2y_2 \end{cases}$.

In matrix form, this is $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \boldsymbol{y}$.

Comment. Hence, we care about systems of differential equations, even if we work with just one function.

Example 64. Write the (third-order) differential equation $y''' = 3y'' - 2y' + y$ as a system of (first-order) differential equations.

Solution. Write $y_1 = y$, $y_2 = y'$ and $y_3 = y''$.

Then, $y''' = 3y'' - 2y' + y$ translates into the first-order system $\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = y_1 - 2y_2 + 3y_3 \end{cases}$.

In matrix form, this is $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -2 & 3 \end{bmatrix} \boldsymbol{y}$.

Example 65. Consider the following system of (second-order) initial value problems:

$$\begin{aligned} y_1'' &= 2y_1' - 3y_2' + 7y_2 \\ y_2'' &= 4y_1' + y_2' - 5y_1 \end{aligned} \qquad y_1(0) = 2,\ y_1'(0) = 3,\ y_2(0) = -1,\ y_2'(0) = 1$$

Write it as a first-order initial value problem in the form $\boldsymbol{y}' = M\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$.

Solution. Introduce $y_3 = y_1'$ and $y_4 = y_2'$. Then the system translates into $\boldsymbol{y}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 7 & 2 & -3 \\ -5 & 0 & 4 & 1 \end{bmatrix} \boldsymbol{y}$, $\boldsymbol{y}(0) = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \end{bmatrix}$.

To solve $\boldsymbol{y}' = M\boldsymbol{y}$, determine the eigenvectors of M.

- Each λ-eigenvector $\boldsymbol{v}$ provides a solution: $\boldsymbol{y}(x) = \boldsymbol{v}e^{\lambda x}$
- If there are enough eigenvectors, these combine to the general solution.

(systems of DEs) The unique solution to $\boldsymbol{y}' = M\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{c}$ is $\boldsymbol{y}(x) = e^{Mx}\boldsymbol{c}$.

- Here, e^{Mx} is the fundamental matrix solution to $\boldsymbol{y}' = M\boldsymbol{y}$, $\boldsymbol{y}(0) = I$ (with I the identity matrix).
- If $\Phi(x)$ is any fundamental matrix solution to $\boldsymbol{y}' = M\boldsymbol{y}$, then $e^{Mx} = \Phi(x)\Phi(0)^{-1}$.
- To construct a fundamental matrix solution $\Phi(x)$, we compute eigenvectors:
  Given a λ-eigenvector $\boldsymbol{v}$, we have the corresponding solution $\boldsymbol{y}(x) = \boldsymbol{v}e^{\lambda x}$.
  If there are enough eigenvectors, we can collect these as columns to obtain $\Phi(x)$.

Note. We are defining the **matrix exponential** e^{Mx} as the solution to an IVP. This is equivalent to how one can define the ordinary exponential e^x as the solution to $y' = y$, $y(0) = 1$.

[In a little bit, we will also discuss how to think about the matrix exponential e^{Mx} using power series.]

Comment. If there are not enough eigenvectors, then (just as in the case of recurrence equations) we know what to do as well (at least in principle): instead of looking only for solutions of the type $\boldsymbol{y}(x) = \boldsymbol{v}e^{\lambda x}$, we also need to look for solutions of the type $\boldsymbol{y}(x) = (\boldsymbol{v}x + \boldsymbol{w})e^{\lambda x}$. Note that this can only happen if an eigenvalue is a repeated root of the characteristic polynomial.

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Important. Compare this to our method of solving systems of REs and for computing matrix powers M^n. Note that the above conclusion about systems of DEs can be deduced along the same lines as what we did for REs:

- If $\Phi(x)$ is a fundamental matrix solution, then so is $\Psi(x) = \Phi(x)C$ for every constant matrix C. (Why?!)
  Therefore, $\Psi(x) = \Phi(x)\Phi(0)^{-1}$ is a fundamental matrix solution with $\Psi(0) = \Phi(0)\Phi(0)^{-1} = I$.
  But e^{Mx} is defined to be the unique such solution, so that $\Psi(x) = e^{Mx}$.
- Let us look for solutions of $\boldsymbol{y}' = M\boldsymbol{y}$ of the form $\boldsymbol{y}(x) = \boldsymbol{v}e^{\lambda x}$. Note that $\boldsymbol{y}' = \lambda\boldsymbol{v}e^{\lambda x} = \lambda\boldsymbol{y}$.
  Plugging into $\boldsymbol{y}' = M\boldsymbol{y}$, we find $\lambda\boldsymbol{y} = M\boldsymbol{y}$.
  In other words, $\boldsymbol{y}(x) = \boldsymbol{v}e^{\lambda x}$ is a solution if and only if $\boldsymbol{v}$ is a λ-eigenvector of M.

Observe how the next example proceeds along the same lines as Example 60.

Example 66. Let $M = \begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix}$.

(a) Determine the general solution to $\boldsymbol{y}' = M\boldsymbol{y}$.

(b) Determine a fundamental matrix solution to $\boldsymbol{y}' = M\boldsymbol{y}$.

(c) Compute e^{Mx}.

(d) Solve the initial value problem $\boldsymbol{y}' = M\boldsymbol{y}$ with $\boldsymbol{y}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Solution.

(a) Recall that each λ-eigenvector $\boldsymbol{v}$ of M provides us with a solution: namely, $\boldsymbol{y}(x) = \boldsymbol{v}e^{\lambda x}$.
We computed earlier that $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 3$, and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = -2$.
Hence, the general solution is $C_1\begin{bmatrix} 2 \\ 1 \end{bmatrix}e^{3x} + C_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{-2x}$.

(b) The corresponding fundamental matrix solution is $\Phi(x) = \begin{bmatrix} 2\cdot e^{3x} & e^{-2x} \\ e^{3x} & e^{-2x} \end{bmatrix}$.
[Note that our general solution is precisely $\Phi(x)\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$.]

(c) Since $\Phi(0) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, we have $\Phi(0)^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$. It follows that

$$e^{Mx} = \Phi(x)\Phi(0)^{-1} = \begin{bmatrix} 2\cdot e^{3x} & e^{-2x} \\ e^{3x} & e^{-2x} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2\cdot e^{3x} - e^{-2x} & -2\cdot e^{3x} + 2e^{-2x} \\ e^{3x} - e^{-2x} & -e^{3x} + 2e^{-2x} \end{bmatrix}.$$

Check. Let us verify the formula for e^{Mx} in the simple case $x = 0$: $e^{M0} = \begin{bmatrix} 2-1 & -2+2 \\ 1-1 & -1+2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(d) The solution to the IVP is $\boldsymbol{y}(x) = e^{Mx}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2\cdot e^{3x} + 2e^{-2x} \\ -e^{3x} + 2e^{-2x} \end{bmatrix}$ (the second column of e^{Mx}).

Sage. We can compute the matrix exponential in Sage as follows:

```
>>> M = matrix([[8,-10],[5,-7]])
```

```
>>> exp(M*x)
```

$$\begin{pmatrix} (2\,e^{(5\,x)} - 1)\,e^{(-2\,x)} & -2\,(e^{(5\,x)} - 1)\,e^{(-2\,x)} \\ (e^{(5\,x)} - 1)\,e^{(-2\,x)} & -(e^{(5\,x)} - 2)\,e^{(-2\,x)} \end{pmatrix}$$

Note that this indeed matches the result of our computation.
[By the way, the variable x is pre-defined as a symbolic variable in Sage. That's why, unlike for n in the computation of M^n, we did not need to use `x = var('x')` first.]