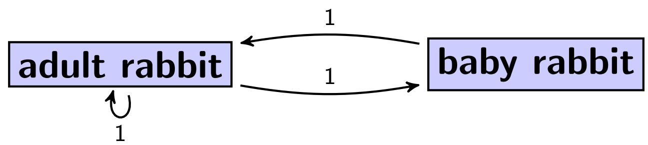
Example 59. We model rabbit reproduction as follows.

Each month, every pair of adult rabbits produces one pair of baby rabbit as offspring. Meanwhile, it takes baby rabbits one month to mature to adults.



Comment. In this simplified model, rabbits always come in male/female pairs and no rabbits die. Though these feautures might make it sound fairly useless, the model may have some merit when describing populations under ideal conditions (unlimited resources) and over short time (no deaths).

Historical comment. The question how many rabbits there are after one year, when starting out with a pair of baby rabbits is famously included in the 1202 textbook of the Italian mathematician Leonardo of Pisa, known as Fibonacci.

If we start with one baby rabbit pair, how many adult rabbits are there after n months?

Solution. Let a_n be the number of adult rabbit pairs after n months. Likewise, b_n is the number of baby rabbit pairs. The transition from one month to the next is given by $a_{n+1} = a_n + b_n$ and $b_{n+1} = a_n$. Using $b_n = a_{n-1}$ (from the second equation) in the first equation, we obtain $a_{n+1} = a_n + a_{n-1}$.

The initial conditions are $a_0 = 0$ and $a_1 = 1$ (the latter follows from $b_0 = 1$).

It follows that the number b_n of adult rabbits are precisely the Fibonacci numbers F_n.

Comment. Note that the transition from one month to the next is described by in matrix-vector form as

$$\begin{bmatrix} a_{n+1} \\ b_{n+1} \end{bmatrix} = \begin{bmatrix} a_n + b_n \\ a_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_n \\ b_n \end{bmatrix}.$$

Writing $\boldsymbol{a}_n = \begin{bmatrix} a_n \\ b_n \end{bmatrix}$, this becomes $\boldsymbol{a}_{n+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \boldsymbol{a}_n$ with $\boldsymbol{a}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Consequently, $\boldsymbol{a}_n = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \boldsymbol{a}_0 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Solving systems of recurrence equations

To solve $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$, determine the eigenvectors of M.

- Each λ-eigenvector $\boldsymbol{v}$ provides a solution: $\boldsymbol{a}_n = \boldsymbol{v}\lambda^n$ [assuming that $\lambda \neq 0$]
- If there are enough eigenvectors, these combine to the general solution.

Why? When solving single linear recurrences, we found that the basic solutions are of the form cr^n where $r \neq 0$ is a root of the characteristic polynomials. To solve $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$, it is therefore natural to look for solutions of the form $\boldsymbol{a}_n = \boldsymbol{c}r^n$ (where $\boldsymbol{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$). Note that $\boldsymbol{a}_{n+1} = \boldsymbol{c}r^{n+1} = r\boldsymbol{a}_n$.

Plugging into $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$ we find $\boldsymbol{c}r^{n+1} = M\boldsymbol{c}r^n$.

Cancelling r^n (just a nonzero number!), this simplifies to $r\boldsymbol{c} = M\boldsymbol{c}$.

In other words, $\boldsymbol{a}_n = \boldsymbol{c}r^n$ is a solution if and only if $\boldsymbol{c}$ is an r-eigenvector of M.

Comment. If there are not enough eigenvectors, then we know what to do as well (at least in principle): instead of looking only for solutions of the type $\boldsymbol{a}_n = \boldsymbol{v}\lambda^n$, we also need to look for solutions of the type $\boldsymbol{a}_n = (\boldsymbol{v}n + \boldsymbol{w})\lambda^n$. Note that this can only happen if an eigenvalue is a repeated root of the characteristic polynomial.

Example 60. Let $M = \begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix}$.

(a) Determine the general solution to $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$.

(b) Determine a **fundamental matrix solution** to $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$.

(c) Compute M^n.

Solution.

(a) Recall that each λ-eigenvector $\boldsymbol{v}$ of M provides us with a solution: $\boldsymbol{a}_n = \boldsymbol{v}\lambda^n$

We computed earlier that $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 3$, and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = -2$.

Hence, the general solution is $C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} 3^n + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} (-2)^n$.

(b) Note that we can write the general solution as

$\boldsymbol{a}_n = C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} 3^n + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} (-2)^n = \begin{bmatrix} 2 \cdot 3^n & (-2)^n \\ 3^n & (-2)^n \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$.

We call $\Phi_n = \begin{bmatrix} 2 \cdot 3^n & (-2)^n \\ 3^n & (-2)^n \end{bmatrix}$ the corresponding **fundamental matrix (solution)**.

Note that our general solution is precisely $\Phi_n \boldsymbol{c}$ with $\boldsymbol{c} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$.

Observations.

(a) The columns of Φ_n are (independent) solutions of the system.

(b) Φ_n solves the RE itself: $\Phi_{n+1} = M\Phi_n$.
[Spell this out in this example! That Φ_n solves the RE follows from the definition of matrix multiplication.]

(c) It follows that $\Phi_n = M^n \Phi_0$. Equivalently, $\Phi_n \Phi_0^{-1} = M^n$. (See next part!)

(c) Note that $\Phi_0 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, so that $\Phi_0^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$. It follows that

$$M^n = \Phi_n \Phi_0^{-1} = \begin{bmatrix} 2 \cdot 3^n & (-2)^n \\ 3^n & (-2)^n \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 \cdot 3^n - (-2)^n & -2 \cdot 3^n + 2(-2)^n \\ 3^n - (-2)^n & -3^n + 2(-2)^n \end{bmatrix}.$$

Check. Let us verify the formula for M^n in the cases $n = 0$ and $n = 1$:

$M^0 = \begin{bmatrix} 2-1 & -2+2 \\ 1-1 & -1+2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$M^1 = \begin{bmatrix} 2 \cdot 3 - (-2) & -2 \cdot 3 + 2(-2) \\ 3 - (-2) & -3 + 2(-2) \end{bmatrix} = \begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix}$

We just saw that being able to compute matrix powers is equivalent to solving systems of recurrences. Indeed, we can use this fact to compute matrix powers.

(a way to compute powers of a matrix M)
Compute a fundamental matrix solution Φ_n of $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$.
Then $M^n = \Phi_n \Phi_0^{-1}$.

If you have taken linear algebra classes, you may have learned that matrix powers M^n can be computed by diagonalizing the matrix M. The latter hinges on computing eigenvalues and eigenvectors of M as well. Compare the two approaches!