

Example 37. We have been factoring differential operators like $D^2 + 4D + 4 = (D + 2)^2$.

Things become much more complicated when the coefficients are not constant!

For instance, the linear DE $y'' + 4y' + 4xy = 0$ can be written as $Ly = 0$ with $L = D^2 + 4D + 4x$. However, in general, such operators cannot be factored (unless we allow as coefficients functions in x that we are not familiar with). [On the other hand, any ordinary polynomial can be factored over the complex numbers.]

One indication that things become much more complicated is that x and D do not commute: $xD \neq Dx$!!

Indeed, $(xD)f(x) = xf'(x)$ while $(Dx)f(x) = \frac{d}{dx}[xf(x)] = f(x) + xf'(x) = (1 + xD)f(x)$.

This computation shows that, in fact, $Dx = xD + 1$.

Review. Linear DEs are those that can be written as $Ly = f(x)$ where L is a linear differential operator: namely,

$$L = p_n(x)D^n + p_{n-1}(x)D^{n-1} + \dots + p_1(x)D + p_0(x). \tag{1}$$

Recall that the operators xD and Dx are not the same: instead, $Dx = xD + 1$.

We say that an operator of the form (1) is in **normal form**.

For instance. xD is in normal form, whereas Dx is not in normal form. It follows from the previous example that the normal form of Dx is $xD + 1$.

Example 38. Let $a = a(x)$ be some function.

- (a) Write the operator Da in normal form [normal form means as in (1)].
- (b) Write the operator D^2a in normal form.

Solution.

(a) $(Da)f(x) = \frac{d}{dx}[a(x)f(x)] = a'(x)f(x) + a(x)f'(x) = (a' + aD)f(x)$

Hence, $Da = aD + a'$.

(b) $(D^2a)f(x) = \frac{d^2}{dx^2}[a(x)f(x)] = \frac{d}{dx}[a'(x)f(x) + a(x)f'(x)] = a''(x)f(x) + 2a'(x)f'(x) + a(x)f''(x)$
 $= (a'' + 2a'D + aD^2)f(x)$

Hence, $D^2a = aD^2 + 2a'D + a''$.

Example 39. Suppose that a and b depend on x . Expand $(D + a)(D + b)$ in normal form.

Solution. $(D + a)(D + b) = D^2 + Db + aD + ab = D^2 + (bD + b') + aD + ab = D^2 + (a + b)D + ab + b'$

Comment. Of course, if b is a constant, then $b' = 0$ and we just get the familiar expansion.

Comment. At this point, it is not surprising that, in general, $(D + a)(D + b) \neq (D + b)(D + a)$.

Example 40. Suppose we want to factor $D^2 + pD + q$ as $(D + a)(D + b)$. [p, q, a, b depend on x]

(a) Spell out equations to find a and b .

(b) Find all factorizations of D^2 . [An obvious one is $D^2 = D \cdot D$ but there are others!]

Solution.

(a) Matching coefficients with $(D + a)(D + b) = D^2 + (a + b)D + ab + b'$, we find that we need

$$p = a + b, \quad q = ab + b'.$$

Equivalently, $a = p - b$ and $q = (p - b)b + b'$. The latter is a nonlinear (!) DE for b . Once solved for b , we obtain a as $a = p - b$.

(b) This is the case $p = q = 0$. The DE for b becomes $b' = b^2$.

Because it is separable (show all details!), we find that $b(x) = \frac{1}{C - x}$ or $b(x) = 0$.

Since $a = -b$, we obtain the factorizations $D^2 = \left(D - \frac{1}{C - x}\right)\left(D + \frac{1}{C - x}\right)$ and $D^2 = D \cdot D$.

Our computations show that there are no further factorizations.

Comment. Note that this example illustrates that factorization of differential operators is not unique!

For instance, $D^2 = D \cdot D$ and $D^2 = \left(D + \frac{1}{x}\right) \cdot \left(D - \frac{1}{x}\right)$ (the case $C = 0$ above).

Comment. In general, the nonlinear DE for b does not have any polynomial or rational solution (or, in fact, any solution that can be expressed in terms of functions that we are familiar with).