Example 23. (review) Find the general solution of $y''' - y'' - 5y' - 3y = 0$.

Solution. The characteristic polynomial $p(D) = D^3 - D^2 - 5D - 3 = (D-3)(D+1)^2$ has roots $3, -1, -1$.
Hence, the general solution is $y(x) = C_1 e^{3x} + (C_2 + C_3 x)e^{-x}$.

Example 24. Find the general solution of $y'' + y = 0$.

Solution. The characteristic polynomials is $p(D) = D^2 + 1 = 0$ which has no solutions over the reals.
Over the **complex numbers**, by definition, the roots are i and $-i$.
So the general solution is $y(x) = C_1 e^{ix} + C_2 e^{-ix}$.

Solution. On the other hand, we easily check that $y_1 = \cos(x)$ and $y_2 = \sin(x)$ are two solutions.
Hence, the general solution can also be written as $y(x) = D_1 \cos(x) + D_2 \sin(x)$.

Important comment. That we have these two different representations is a consequence of **Euler's identity**

$$e^{ix} = \cos(x) + i \sin(x).$$

Note that $e^{-ix} = \cos(x) - i\sin(x)$.
On the other hand, $\cos(x) = \frac{1}{2}(e^{ix} + e^{-ix})$ and $\sin(x) = \frac{1}{2i}(e^{ix} - e^{-ix})$.
[Recall that the first formula is an instance of $\mathrm{Re}(z) = \frac{1}{2}(z + \bar{z})$ and the second of $\mathrm{Im}(z) = \frac{1}{2i}(z - \bar{z})$.]

Example 25. Find the general solution of $y'' - 4y' + 13y = 0$.

Solution. The characteristic polynomial $p(D) = D^2 - 4D + 13$ has roots $2 + 3i, 2 - 3i$.
Hence, the general solution is $y(x) = C_1 e^{2x}\cos(3x) + C_2 e^{2x}\sin(3x)$.
Note. $e^{(2+3i)x} = e^{2x}e^{3ix} = e^{2x}(\cos(3x) + i\sin(3x))$

Inhomogeneous linear DEs with constant coefficients

Example 26. Find the general solution of $y'' + 4y = 12x$.

Solution. Here, $p(D) = D^2 + 4$, which has roots $\pm 2i$.
Hence, the general solution is $y(x) = y_p(x) + C_1\cos(2x) + C_2\sin(2x)$. It remains to find a particular solution y_p.
Noting that $D^2 \cdot (12x) = 0$, we apply D^2 to both sides of the DE.
We get $D^2(D^2 + 4) \cdot y = 0$, which is a homogeneous linear DE! Its general solution is $C_1 + C_2 x + C_3\cos(2x) + C_4\sin(2x)$. In particular, y_p is of this form for some choice of $C_1, \ldots, C_4$.
It simplifies our life to note that there has to be a particular solution of the simpler form $y_p = C_1 + C_2 x$.
[Why?! Because we know that $C_3\cos(2x) + C_4\sin(2x)$ can be added to any particular solution.]
It only remains to find appropriate values C_1, C_2 such that $y_p'' + 4y_p = 12x$. Since $y_p'' + 4y_p = 4C_1 + 4C_2 x$, comparing coefficients yields $4C_1 = 0$ and $4C_2 = 12$, so that $C_1 = 0$ and $C_2 = 3$. In other words, $y_p = 3x$.
Therefore, the general solution to the original DE is $y(x) = 3x + C_1\cos(2x) + C_2\sin(2x)$.

Example 27. Find the general solution of $y'' + 4y' + 4y = e^{3x}$.

Solution. This is $p(D)y = e^{3x}$ with $p(D) = D^2 + 4D + 4 = (D+2)^2$.
Hence, the general solution is $y(x) = y_p(x) + (C_1 + C_2 x)e^{-2x}$. It remains to find a particular solution y_p.
Note that $(D-3)e^{3x} = 0$. Hence, we apply $(D-3)$ to the DE to get $(D-3)(D+2)^2 y = 0$.
This homogeneous linear DE has general solution $(C_1 + C_2 x)e^{-2x} + C_3 e^{3x}$. We conclude that the original DE must have a particular solution of the form $y_p = C_3 e^{3x}$.
To determine the value of C_3, we plug into the original DE: $y_p'' + 4y_p' + 4y_p = (9 + 4 \cdot 3 + 4)C_3 e^{3x} \overset{!}{=} e^{3x}$. Hence, $C_3 = 1/25$. In conclusion, the general solution is $y(x) = (C_1 + C_2 x)e^{-2x} + \frac{1}{25}e^{3x}$.

Armin Straub
straub@southalabama.edu

We found a recipe for solving nonhomogeneous linear DEs with constant coefficients.

Our approach works for $p(D)y = f(x)$ whenever the right-hand side $f(x)$ is the solution of some homogeneous linear DE with constant coefficients: $q(D)f(x) = 0$

Theorem 28. (method of undetermined coefficients) To find a particular solution y_p to an inhomogeneous linear DE with constant coefficients $p(D)y = f(x)$:

- Find $q(D)$ so that $q(D)f(x) = 0$. [This does not work for all $f(x)$.]
- Let $r_1, \ldots, r_n$ be the ("old") roots of the polynomial $p(D)$.
  Let $s_1, \ldots, s_m$ be the ("new") roots of the polynomial $q(D)$.
- It follows that y_p solves the **homogeneous** DE $q(D)\,p(D)y = 0$.

  The characteristic polynomial of this DE has roots $r_1, \ldots, r_n, s_1, \ldots, s_m$.

  Let $v_1, \ldots, v_m$ be the "new" solutions (i.e. not solutions of the "old" $p(D)y = 0$).

  By plugging into $p(D)y_p = f(x)$, we find (unique) C_i so that $y_p = C_1 v_1 + \ldots + C_m v_m$.

Because of the final step, this approach is often called **method of undetermined coefficients**.

For which $f(x)$ does this work? By Theorem 20, we know exactly which $f(x)$ are solutions to homogeneous linear DEs with constant coefficients: these are linear combinations of exponentials $x^j\, e^{rx}$ (which includes $x^j\, e^{ax}\cos(bx)$ and $x^j\, e^{ax}\sin(bx)$).

Example 29. (again) Determine the general solution of $y'' + 4y' + 4y = e^{3x}$.

Solution. The "old" roots are $-2, -2$. The "new" roots are 3. Hence, there has to be a particular solution of the form $y_p = Ce^{3x}$. To find the value of C, we plug into the DE.

$y_p'' + 4y_p' + 4y_p = (9 + 4\cdot 3 + 4)Ce^{3x} \overset{!}{=} e^{3x}$. Hence, $C = 1/25$.

Therefore, the general solution is $y(x) = (C_1 + C_2 x)e^{-2x} + \frac{1}{25}e^{3x}$.