Review: Linear first-order DEs

The most general first-order linear DE is $y' = a(x)y + f(x)$.

We will recall next time that we can always solve it.

The corresponding **homogeneous** linear DE is $y' = a(x)y$.

Important comment. Write $D = \frac{\mathrm{d}}{\mathrm{d}x}$. Then we can write $y' - a(x)y = f(x)$ as $Ly = f(x)$ where $L = D - a(x)$. The corresponding homogeneous DE is simply $Ly = 0$.

Solving linear first-order DEs using variation of constants

The following DE is linear and first-order (but not with constant coefficients).

Example 13. Solve $y' = x^2 y$.

Solution. This DE is separable as well: $\frac{1}{y}\mathrm{d}y = x^2\,\mathrm{d}x$ (note that we just lost the solution $y = 0$).

Integrating gives $\ln|y| = \frac{1}{3}x^3 + A$, so that $|y| = e^{\frac{1}{3}x^2 + A}$. Since the RHS is never zero, we must have either $y = e^{\frac{1}{3}x^2 + A}$ or $y = -e^{\frac{1}{3}x^2 + A}$.

Hence $y = \pm e^A e^{\frac{1}{3}x^3} = Ce^{\frac{1}{3}x^3}$ (with $C = \pm e^A$). Note that $C = 0$ corresponds to the singular solution $y = 0$.

In summary, the general solution is $y = Ce^{\frac{1}{3}x^3}$ (with C any real number).

Check. Compute y' and verify that the DE is indeed satisfied.

As in the previous example, we can immediately solve any **homogeneous** linear first-order DE:

Example 14. Solve $y' = a(x)y$.

Solution. Proceeding as in the previous example, we find $y(x) = Ce^{\int a(x)\mathrm{d}x}$.

Check. Compute y' and verify that the DE is indeed satisfied.

Recall that, to find the general solution of the inhomogeneous DE $y' = a(x)y + f(x)$, we only need to find a particular solution y_p.

Then the general solution is $y_p + y_h$, where y_h is the general solution of the homogeneous DE $y' = a(x)y$.

Theorem 15. (variation of constants) $y' = a(x)y + f(x)$ has the particular solution

$$y_p(x) = y_h(x)\int \frac{f(x)}{y_h(x)}\mathrm{d}x,$$

where $y_h(x) = e^{\int a(x)\mathrm{d}x}$ is any solution to the homogeneous equation $y' = a(x)y$.

Proof. $y_p'(x) = y_h'(x)\int \frac{f(x)}{y_h(x)}\mathrm{d}x + y_h(x)\underbrace{\frac{\mathrm{d}}{\mathrm{d}x}\int \frac{f(x)}{y_h(x)}\mathrm{d}x}_{\frac{f(x)}{y_h(x)}} = a(x)y_h(x)\int \frac{f(x)}{y_h(x)}\mathrm{d}x + f(x) = a(x)y + f(x)$ □

Comment. Note that the formula for $y_p(x)$ gives the general solution if we let $\int \frac{f(x)}{y_h(x)}\mathrm{d}x$ be the general antiderivative. (Think about the effect of the constant of integration!)

Armin Straub
straub@southalabama.edu

Recall. The formula for $y_p(x)$ can be found using **variation of constants** (sometimes called variation of parameters): that is, we look for solutions of the form $y(x) = c(x) y_h(x)$.

If we plug $y(x) = c(x) y_h(x)$ into the DE, we find $c' y_h + c y_h' = a c y_h + f$. Since $y_h' = a y_h$, this simplifies to $c' y_h = f$ or, equivalently, $c' = \frac{f}{y_h}$. Hence, $c(x) = \int \frac{f(x)}{y_h(x)} \mathrm{d}x$, which is the formula in the theorem.

Example 16. Solve $x^2 y' = 1 - x y + 2x$, $y(1) = 3$.

Solution. Write as $\frac{\mathrm{d}y}{\mathrm{d}x} = a(x) y + f(x)$ with $a(x) = -\frac{1}{x}$ and $f(x) = \frac{1}{x^2} + \frac{2}{x}$.

$y_h(x) = e^{\int a(x) \mathrm{d}x} = e^{-\ln x} = \frac{1}{x}$. (Why can we write $\ln x$ instead of $\ln|x|$?!) Hence:

$$y_p(x) = y_h(x) \int \frac{f(x)}{y_h(x)} \mathrm{d}x = \frac{1}{x} \int \left(\frac{1}{x} + 2\right) \mathrm{d}x = \frac{\ln x + 2x + C}{x}$$

Using $y(1) = 3$, we find $C = 1$. In summary, the solution is $y = \frac{\ln(x) + 2x + 1}{x}$.

Comment. Observe how the general solution (with parameter C) is indeed obtained from any particular solution (say, $\frac{\ln x + 2x}{x}$) plus the general solution to the homogeneous equation, which is $\frac{C}{x}$.

Armin Straub
straub@southalabama.edu