

Example 9. (review) $[1 \ 2 \ 3] \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = [14]$ whereas $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} [1 \ 2 \ 3] = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$.

Review: Examples of differential equations we can solve

Let's start with one of the simplest (and most fundamental) differential equations (DE). It is **first-order** (only a first derivative) and **linear** (with constant coefficients).

Example 10. Solve $y' = 3y$.

Solution. $y(x) = Ce^{3x}$

Check. Indeed, if $y(x) = Ce^{3x}$, then $y'(x) = 3Ce^{3x} = 3y(x)$.

Comment. Recall we can always easily check whether a function solves a differential equation. This means that (although you might be unfamiliar with the techniques for solving) you can use computer algebra systems like Sage to solve differential equations without trust issues.

To describe a unique solution, additional constraints need to be imposed.

Example 11. Solve the **initial value problem** (IVP) $y' = 3y$, $y(0) = 5$.

Solution. This has the unique solution $y(x) = 5e^{3x}$.

The following is a **non-linear** differential equation. In general, such equations are much more complicated than linear ones. We can solve this particular one because it is **separable**.

Example 12. Solve $y' = xy^2$.

Solution. This DE is separable: $\frac{1}{y^2}dy = x dx$. Integrating, we find $-\frac{1}{y} = \frac{1}{2}x^2 + C$.

Hence, $y = -\frac{1}{\frac{1}{2}x^2 + C} = \frac{2}{D - x^2}$.

[Here, $D = -2C$ but that relationship doesn't matter; it only matters that the solution has a free parameter.]

Note. Note that we did not find the solution $y = 0$ (lost when dividing by y^2). It is called a **singular solution** because it is not part of the **general solution** (the one-parameter family found above). [Although, we can obtain it from the general solution by letting $D \rightarrow \infty$.]

Check. Compute y' and verify that the DE is indeed satisfied.