Example 162. Find the unique solution $u(x,t)$ to: $\begin{array}{l} u_t = u_{xx} \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = 1, \quad x \in (0,1) \end{array}$

Solution. This is the case $k=1$, $L=1$ and $f(x)=1$, $x \in (0,1)$, of Example 160.

In the final step, we extend $f(x)$ to the 2-periodic odd function of Example 139. In particular, earlier, we have already computed that the Fourier series is

$$f(x) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \sin(n\pi x).$$

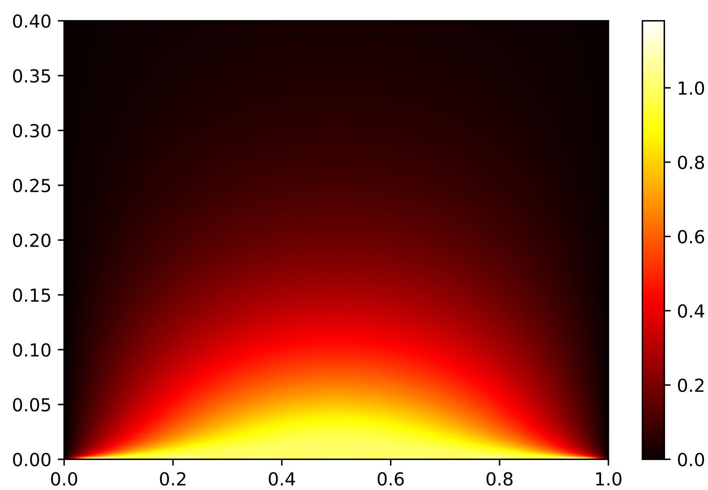
Hence, $u(x,t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} e^{-\pi^2 n^2 t} \sin(n\pi x)$.

Comment. Note that, for $t > 0$, the exponential very quickly approaches 0 (because of the $-n^2$ in the exponent), so that we get very accurate approximations with only a handful terms.

We can use Sage to plot our solution using the terms $n = 1, 3, 5, \ldots, 19$ of the infinite sum:

```
>>> var('x,t');
>>> uxt = sum(4/(pi*n) * exp(-pi^2*n^2*t) * sin(pi*n*x) for n in range(1,20,2))
>>> density_plot(uxt, (x,0,1), (t,0,0.4), plot_points=200, cmap='hot')
```

The resulting plot should look similar to the following:



Can you make sense of the plot? Does that plot confirm our expectations?

[Note that the horizontal axis shows x for $x \in (0, 1)$, while the vertical axis shows t for $t \in (0, 0.4)$. Yellow represents 1 (for $t = 0$, all values are 1 because of the initial condition), while black represents 0.]

The boundary conditions in the next example model insulated ends.

Observe how we can proceed exactly as in Example 160.

Example 163. Find the unique solution $u(x,t)$ to:

$$\begin{aligned} u_t &= k u_{xx} && \text{(PDE)} \\ u_x(0,t) &= u_x(L,t) = 0 && \text{(BC)} \\ u(x,0) &= f(x), \quad x \in (0,L) && \text{(IC)} \end{aligned}$$

Solution.

- We proceed as before and look for solutions $u(x,t) = X(x)T(t)$ (**separation of variables**).

  Plugging into (PDE), we get $X(x)T'(t) = kX''(x)T(t)$, and so $\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)} = \text{const} =: -\lambda$.
  We thus have $X'' + \lambda X = 0$ and $T' + \lambda k T = 0$.

- From the (BC), i.e. $u_x(0,t) = X'(0)T(t) = 0$, we get $X'(0) = 0$.
  Likewise, $u_x(L,t) = X'(L)T(t) = 0$ implies $X'(L) = 0$.

- So X solves $X'' + \lambda X = 0$, $X'(0) = 0$, $X'(L) = 0$. It is shown in Example 156 that, up to multiples, the only nonzero solutions of this eigenvalue problem are $X(x) = \cos(\frac{\pi n}{L} x)$ corresponding to $\lambda = (\frac{\pi n}{L})^2$, $n = 0, 1, 2, 3....$

- On the other hand (as before), T solves $T' + \lambda k T = 0$, and hence $T(t) = e^{-\lambda k t} = e^{-(\frac{\pi n}{L})^2 k t}$.

- Taken together, we have the solutions $u_n(x,t) = e^{-(\frac{\pi n}{L})^2 k t} \cos(\frac{\pi n}{L} x)$ solving (PDE)+(BC).

- We wish to combine these in such a way that (IC) holds as well.
  At $t = 0$, $u_n(x,0) = \cos(\frac{\pi n}{L} x)$. All of these are $2L$-periodic.

  Hence, we extend $f(x)$, which is only given on $(0,L)$, to an even $2L$-periodic function (its Fourier cosine series!). By making it even, its Fourier series only involves cosine terms: $f(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos(\frac{\pi n}{L} x)$.
  Note that

  $$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x,$$

  where the first integral makes reference to the extension of $f(x)$ while the second integral only uses $f(x)$ on its original interval of definition.

Consequently, (PDE)+(BC)+(IC) is solved by

$$u(x,t) = \frac{a_0}{2} u_0(x,t) + \sum_{n=1}^{\infty} a_n u_n(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-(\frac{\pi n}{L})^2 k t} \cos\left(\frac{\pi n}{L} x\right),$$

where

$$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x.$$

Example 164. Find the unique solution $u(x,t)$ to:

$$\begin{aligned} u_t &= 3 u_{xx} && \text{(PDE)} \\ u_x(0,t) &= u_x(4,t) = 0 && \text{(BC)} \\ u(x,0) &= 2 + 5\cos(\pi x) - \cos(3\pi x), \ x \in (0,4) && \text{(IC)} \end{aligned}$$

Solution. This is the case $k = 3$, $L = 4$ that we solved in Example 163 where we found that the functions

$$u_n(x,t) = e^{-(\frac{\pi n}{L})^2 k t} \cos\left(\frac{\pi n}{L} x\right) = e^{-3(\frac{\pi n}{4})^2 t} \cos\left(\frac{\pi n}{4} x\right)$$

solve (PDE)+(BC). Since $u_n(x,0) = \cos(\frac{\pi n}{4} x)$, we have

$$2u_0(x,0) + 5u_4(x,0) - u_{12}(x,0) = 2 + 5\cos(\pi x) - \cos(3\pi x),$$

which is what we need for the right-hand side of (IC). Therefore, (PDE)+(BC)+(IC) is solved by

$$u(x,t) = 2u_0(x,t) + 5u_4(x,t) - u_{12}(x,t) = 2 + 5e^{-3\pi^2 t} \cos(\pi x) - e^{-27\pi^2 t} \cos(3\pi x).$$

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