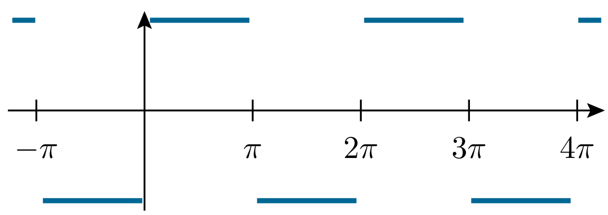
Example 137. Find the Fourier series of the 2π-periodic function $f(t)$ defined by

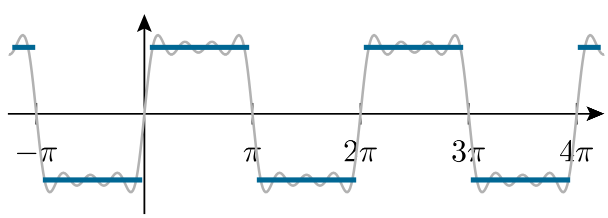
$$f(t) = \begin{cases} -1, & \text{for } t \in (-\pi, 0), \\ +1, & \text{for } t \in (0, \pi), \\ 0, & \text{for } t = -\pi, 0, \pi. \end{cases}$$



Solution. We compute $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\mathrm{d}t = 0$, as well as

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\mathrm{d}t = \frac{1}{\pi}\left[-\int_{-\pi}^{0}\cos(nt)\mathrm{d}t + \int_0^{\pi}\cos(nt)\mathrm{d}t\right] = 0 \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\mathrm{d}t = \frac{1}{\pi}\left[-\int_{-\pi}^{0}\sin(nt)\mathrm{d}t + \int_0^{\pi}\sin(nt)\mathrm{d}t\right] = \frac{2}{\pi n}[1 - \cos(n\pi)] \\ &= \frac{2}{\pi n}[1 - (-1)^n] = \begin{cases} \frac{4}{\pi n} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}. \end{aligned}$$

In conclusion, $f(t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n}\sin(nt) = \frac{4}{\pi}\left(\sin(t) + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t) + \ldots\right)$.



Observation. The coefficients a_n are zero for all n if and only if $f(t)$ is odd.

Comment. The value of $f(t)$ for $t = -\pi, 0, \pi$ is irrelevant to the computation of the Fourier series. They are chosen so that $f(t)$ is equal to the Fourier series for all t (recall that, at a jump discontinuity t, the Fourier series converges to the average $\frac{f(t^-) + f(t^+)}{2}$).

Comment. Plot the (sum of the) first few terms of the Fourier series. What do you observe? The "overshooting" is known as the **Gibbs phenomenon**: `https://en.wikipedia.org/wiki/Gibbs_phenomenon`

Comment. Set $t = \frac{\pi}{2}$ in the Fourier series we just computed, to get Leibniz' series $\pi = 4[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots]$. For such an alternating series, the error made by stopping at the term $1/n$ is on the order of $1/n$. To compute the 768 digits of π to get to the Feynman point ($3.14159265\ldots721134999999\ldots$), we would (roughly) need $1/n < 10^{-768}$, or $n > 10^{768}$. That's a lot of terms! (Roger Penrose, for instance, estimates that there are about 10^{80} atoms in the observable universe.)

Remark. Convergence of such series is not completely obvious! Recall, for instance, that the (odd part of) the harmonic series $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$ diverges. (On the other hand, do you remember the alternating sign test from Calculus II?)

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Fourier series with general period

There is nothing special about 2π-periodic functions considered before (except that $\cos(t)$ and $\sin(t)$ have fundamental period 2π). Note that $\cos(\pi t/L)$ and $\sin(\pi t/L)$ have period $2L$.

Theorem 138. Every* $2L$-periodic function f can be written as a **Fourier series**

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right) \right).$$

Technical detail*: f needs to be, e.g., piecewise smooth.
Also, if t is a discontinuity, then the Fourier series converges to the average $\frac{f(t^-) + f(t^+)}{2}$.

The **Fourier coefficients** a_n, b_n are unique and can be computed as

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t) \cos\left(\frac{n\pi t}{L}\right) \mathrm{d}t, \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(t) \sin\left(\frac{n\pi t}{L}\right) \mathrm{d}t.$$

Comment. This follows from Theorem 133 because, if $f(t)$ has period $2L$, then $\tilde{f}(t) := f\left(\frac{L}{\pi}t\right)$ has period 2π.

Example 139. Find the Fourier series of the 2-periodic function $g(t) = \begin{cases} -1 & \text{for } t \in (-1, 0) \\ +1 & \text{for } t \in (0, 1) \\ 0 & \text{for } t = -1, 0, 1 \end{cases}$.

Solution. Instead of computing from scratch, we can use the fact that $g(t) = f(\pi t)$, with f as in the previous example, to get $g(t) = f(\pi t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \sin(n\pi t)$.

Fourier cosine series and Fourier sine series

Suppose we have a function $f(t)$ which is defined on a finite interval $[0, L]$. Depending on the kind of application, we can extend $f(t)$ to a periodic function in three natural ways; in each case, we can then compute a Fourier series for $f(t)$ (which will agree with $f(t)$ on $[0, L]$).

Comment. Here, we do not worry about the definition of $f(t)$ at specific individual points like $t = 0$ and $t = L$, or at jump discontinuities. Recall that, at a discontinuity, a Fourier series takes the average value.

(a) We can extend $f(t)$ to an L-periodic function.

In that case, we obtain the Fourier series $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{2\pi n t}{L}\right) + b_n \sin\left(\frac{2\pi n t}{L}\right) \right)$.

(b) We can extend $f(t)$ to an even $2L$-periodic function.

In that case, we obtain the **Fourier cosine series** $f(t) = \frac{\tilde{a}_0}{2} + \sum_{n=1}^{\infty} \tilde{a}_n \cos\left(\frac{\pi n t}{L}\right)$.

(c) We can extend $f(t)$ to an odd $2L$-periodic function.

In that case, we obtain the **Fourier sine series** $f(t) = \sum_{n=1}^{\infty} \tilde{b}_n \sin\left(\frac{\pi n t}{L}\right)$.

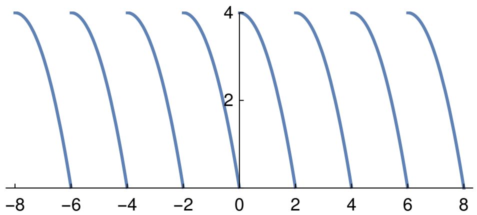
Example 140. Consider the function $f(t) = 4 - t^2$, defined for $t \in [0, 2]$.

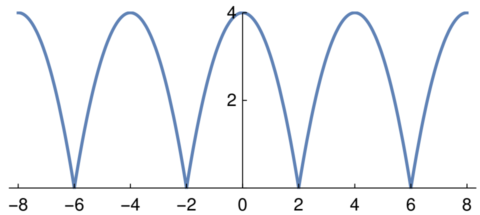
(a) Sketch the 2-periodic extension of $f(t)$.

(b) Sketch the 4-periodic even extension of $f(t)$.

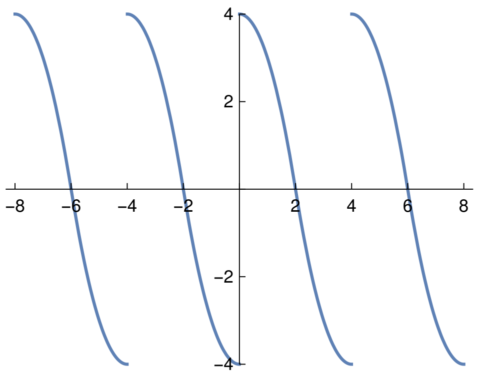
(c) Sketch the 4-periodic odd extension of $f(t)$.

Solution. The 2-periodic extension as well as the 4-periodic even extension:





The 4-periodic odd extension:



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