Review. If $y(x)$ is "nice" at $x = x_0$ (i.e. analytic around $x = x_0$), then

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad \text{with} \quad a_n = \frac{y^{(n)}(x_0)}{n!}.$$

In particular, at $x = 0$,

$$y(x) = \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!} x^n = y(0) + y'(0)x + \frac{1}{2}y''(0)x^2 + \frac{1}{6}y'''(0)x^3 + \ldots$$

Notation. When working with power series $\sum_{n=0}^{\infty} a_n x^n$, we sometimes write $O(x^n)$ to indicate that we omit terms that are multiples of x^n:

For instance. $e^x = 1 + x + \frac{1}{2}x^2 + O(x^3)$ or $\cos(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + O(x^6)$.

Example 110. Let $y(x)$ be the unique solution to the IVP $y' = x^2 + y^2$, $y(0) = 1$.
Determine the first several terms (up to x^4) in the power series of $y(x)$.

Solution. (successive differentiation—for humans) From the DE, $y'(0) = 0^2 + y(0)^2 = 1$.
Differentiating both sides of the DE, we obtain $y'' = 2x + 2yy'$. In particular, $y''(0) = 2$.
Continuing, $y''' = 2 + 2(y')^2 + 2yy''$ so that $y'''(0) = 2 + 2 + 2 \cdot 2 = 8$.
Likewise, $y^{(4)} = 6y'y'' + 2yy'''$ so that $y^{(4)}(0) = 12 + 16 = 28$.
Hence, $y(x) = y(0) + y'(0)x + \frac{1}{2}y''(0)x^2 + \frac{1}{6}y'''(0)x^3 + \frac{1}{24}y^{(4)}(0)x^4 + \ldots = 1 + x + x^2 + \frac{4}{3}x^3 + \frac{7}{6}x^4 + \ldots$

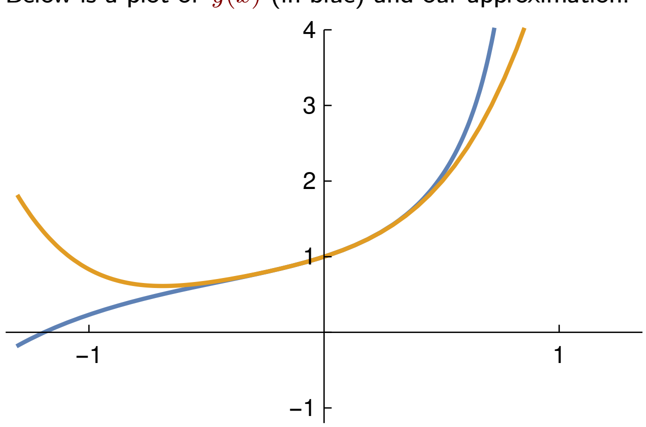
Comment. This approach requires the (symbolic) computation of intermediate derivatives. This is costly (even just the size of the simplified formulas is quickly increasing) and so the solution below is usually preferable for practical purposes. However, successive differentiation works well when working by hand.

Solution. (plug in power series—for computers) The powers series $y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots$ simplifies to $y = 1 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots$ because of the initial condition.
Therefore, $y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots$
To determine a_2, a_3, a_4, a_5, we need to expand $x^2 + y^2$ into a power series:

$y^2 = 1 + 2a_1x + (2a_2 + a_1^2)x^2 + (2a_3 + 2a_1a_2)x^3 + (2a_4 + 2a_1a_3 + a_2^2)x^4 + \ldots$ [we don't need the last term]

Equating coefficients of y' and $x^2 + y^2$, we find $a_1 = 1$, $2a_2 = 2a_1$, $3a_3 = 1 + 2a_2 + a_1^2$, $4a_4 = 2a_3 + 2a_1a_2$.
So $a_1 = 1$, $a_2 = 1$, $a_3 = \frac{4}{3}$, $a_4 = \frac{7}{6}$ and, hence, $y(x) = 1 + x + x^2 + \frac{4}{3}x^3 + \frac{7}{6}x^4 + \ldots$

Below is a plot of $y(x)$ (in blue) and our approximation:



Note how the approximation is very good close to 0 but does not provide us with a "global picture".

Example 111. Let $y(x)$ be the unique solution to the IVP $y''=\cos(x+y)$, $y(0)=0$, $y'(0)=1$. Determine the first several terms (up to x^5) in the power series of $y(x)$.

Solution. (successive differentiation—for humans) From the DE, $y''(0)=\cos(0+y(0))=1$.
Differentiating both sides of the DE, we obtain $y'''=-\sin(x+y)(1+y')$.
In particular, $y'''(0)=-\sin(0+y(0))(1+y'(0))=0$.
Likewise, $y^{(4)}=-\cos(x+y)(1+y')^2-\sin(x+y)y''$ shows $y^{(4)}(0)=-1\cdot 2^2-0=-4$.
Continuing, $y^{(5)}=\sin(x+y)(1+y')^3-3\cos(x+y)(1+y')y''-\sin(x+y)y'''$ so that $y^{(5)}(0)=-6$.
Hence, $y(x)=x+\frac{1}{2}y''(0)x^2+\frac{1}{6}y'''(0)x^3+\frac{1}{24}y^{(4)}(0)x^4+\frac{1}{120}y^{(5)}(0)x^5+\ldots=x+\frac{1}{2}x^2-\frac{1}{6}x^4-\frac{1}{20}x^5+\ldots$

Solution. (plug in power series—for computers) The powers series $y=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\ldots$ simplifies to $y=x+a_2x^2+a_3x^3+a_4x^4+\ldots$ because of the initial conditions.
Therefore, $y'=1+2a_2x+3a_3x^2+4a_4x^3+\ldots$ and $y''=2a_2+6a_3x+12a_4x^2+20a_5x^3+\ldots$
To determine a_2,a_3,a_4,a_5, we need to expand $\cos(x+y)$ into a power series:

Recall that $\cos(x)=1-\frac{1}{2}x^2+\frac{1}{24}x^4+\ldots$

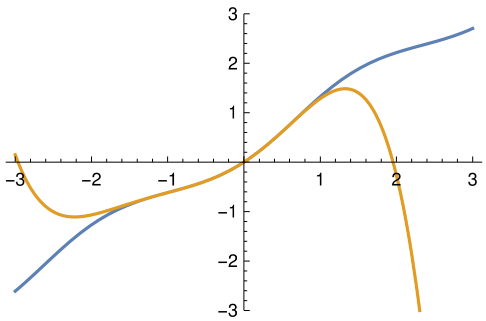
Hence, $\cos(x+y)=1-\frac{1}{2}(x+y)^2+\frac{1}{24}(x+y)^4+\ldots=1-\frac{1}{2}x^2-xy-\frac{1}{2}y^2+O(x^4)$.

Since $y^2=(x+a_2x^2+a_3x^3+\ldots)^2=x^2+2a_2x^3+O(x^4)$,

$\cos(x+y)=1-\frac{1}{2}x^2-x(x+a_2x^2)-\frac{1}{2}(x^2+2a_2x^3)+O(x^4)=1-2x^2-2a_2x^3+O(x^4)$.

Equating coefficients of y'' and $\cos(x+y)$, we find $2a_2=1$, $6a_3=0$, $12a_4=-2$, $20a_5=-2a_2$.
So $a_2=\frac{1}{2}$, $a_3=0$, $a_4=-\frac{1}{6}$, $a_5=-\frac{1}{20}$ and, hence, $y(x)=x+\frac{1}{2}x^2-\frac{1}{6}x^4-\frac{1}{20}x^5+\ldots$

Below is a plot of $y(x)$ (in blue) and our approximation:



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Power series solutions to linear DEs

Note how in the last two examples the "plug in power series" approach was complicated by the fact that the DE was not linear (we had to expand y^2 as well as $\cos(x+y)$, respectively).

For linear DEs, this complication does not arise and we can readily determine the complete power series expansion of analytic solutions (with a recursive description of the coefficients).

Example 112. (Airy equation, part II) Let $y(x)$ be the unique solution to the IVP $y''=xy$, $y(0)=a$, $y'(0)=b$. Determine the power series of $y(x)$.

Solution. (plug in power series) Let us spell out the power series for y, y', y'' and xy:

$$y(x)=\sum_{n=0}^{\infty} a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty} n a_n x^{n-1}=\sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

$$xy(x)=\sum_{n=0}^{\infty} a_n x^{n+1}=\sum_{n=1}^{\infty} a_{n-1}x^n$$

Hence, $y''=xy$ becomes $\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n=\sum_{n=1}^{\infty} a_{n-1}x^n$. We compare coefficients of x^n:

- $n=0$: $2\cdot 1a_2=0$, so that $a_2=0$.
- $n\geqslant 1$: $(n+2)(n+1)a_{n+2}=a_{n-1}$
  Replacing n by $n-2$, this is equivalent to $n(n-1)a_n=a_{n-3}$ for $n\geqslant 3$.

In conclusion, $y(x)=\sum_{n=0}^{\infty} a_n x^n$ with $a_0=a$, $a_1=b$, $a_2=0$ as well as, for $n\geqslant 3$, $a_n=\frac{1}{n(n-1)}a_{n-3}$.

First few terms. In particular, $y=a\left(1+\frac{x^3}{2\cdot 3}+\frac{x^6}{(2\cdot 3)(5\cdot 6)}+\ldots\right)+b\left(x+\frac{x^4}{3\cdot 4}+\frac{x^7}{(3\cdot 4)(6\cdot 7)}+\ldots\right)$.

Advanced. The solution with $y(0)=\frac{1}{3^{2/3}\Gamma(2/3)}$ and $y'(0)=-\frac{1}{3^{1/3}\Gamma(1/3)}$ is known as the **Airy function** $\mathrm{Ai}(x)$.
[A more natural property of $\mathrm{Ai}(x)$ is that it satisfies $y(x)\to 0$ as $x\to\infty$.]