Some special functions and the power series method

Review: power series

Definition 103. A function $y(x)$ is analytic around $x = x_0$ if it has a power series

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

Note. In the next theorem, we will see that this power series is the Taylor series of $y(x)$ around $x = x_0$.

Power series are very pleasant to work with because they behave just like polynomials. For instance, we can differentiate and integrate them:

- If $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$, then $y'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}$ (another power series!).

  We can rewrite the series as $y'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} (x - x_0)^n$.

  The result is a power series just like the one we started with. Likewise, for higher derivatives.

- $\int y(x) \mathrm{d}x = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x - x_0)^{n+1} + C$

Theorem 104. If $y(x)$ is analytic around $x = x_0$, then $y(x)$ is infinitely differentiable and

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad \text{with} \quad a_n = \frac{y^{(n)}(x_0)}{n!}.$$

Caution. Analyticity is needed in this theorem; being infinitely differentiable is not enough. For instance, $y(x) = e^{-1/x^2}$ is infinitely differentiable around $x = 0$ (and everywhere else). However, $y^{(n)}(0) = 0$ for all n.

In particular, if $y(x)$ is analytic at $x = 0$, then

$$y(x) = \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!} x^n = y(0) + y'(0)x + \frac{1}{2} y''(0) x^2 + \frac{1}{6} y'''(0) x^3 + \ldots$$

We have already seen the following example.

Example 105. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{1}{2} x^2 + \frac{1}{3!} x^3 + \ldots$

Once again, notice how the power series clearly has the property that $y' = y$ (as well as $y(0) = 1$).

It follows from here that, for instance, $e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = 1 + 2x + 2x^2 + \frac{4}{3} x^3 + \ldots$

Example 106. Determine the power series for $7e^{3x}$ (at $x = 0$).

Solution. Instead of starting from scratch, we can use that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ to conclude that

$$7e^{3x} = 7 \sum_{n=0}^{\infty} \frac{(3x)^n}{n!} = \sum_{n=0}^{\infty} \frac{7 \cdot 3^n}{n!} x^n = 7 + 21x + \frac{63}{2} x^2 + \frac{63}{2} x^3 + \frac{189}{8} x^4 + \ldots$$

Example 107. Determine the power series for $\cos(x)$ at $x=0$.

Solution. Let $y(x)=\cos(x)$. After computing a few derivatives, we realize that $y^{(2n)}(x)=(-1)^n\cos(x)$ and $y^{(2n+1)}(x)=-(-1)^n\sin(x)$. In particular, $y^{(2n)}(0)=(-1)^n$ and $y^{(2n+1)}(0)=0$. It follows that

$$\cos(x)=\sum_{n=0}^{\infty}\frac{y^{(n)}(0)}{n!}x^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}=1-\frac{x^2}{2}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots$$

Comment. Note that the above observations on $y^{(2n)}$ and $y^{(2n+1)}$ simply reflect the fact that $\cos(x)$ is the unique solution to the IVP $y''=-y$, $y(0)=1$, $y'(0)=0$.

Alternatively. We can also deduce the power series via Euler's formula: $e^{ix}=\cos(x)+i\sin(x)$. Since

$$e^{ix}=\sum_{n=0}^{\infty}\frac{(ix)^n}{n!}=\sum_{m=0}^{\infty}\frac{(ix)^{2m}}{(2m)!}+\sum_{m=0}^{\infty}\frac{(ix)^{2m+1}}{(2m+1)!}=\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{(2m)!}+i\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+1}}{(2m+1)!},$$

we conclude that $\cos(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}$ and $\sin(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$.

Example 108. Determine the first several terms in the power series of $\sin(2x^3)$ at $x=0$.

Solution. (direct—unpleasant) If $f(x)=\sin(2x^3)$, then $f'(x)=6x^2\cos(2x^3)$ as well as $f''(x)=12x\cos(2x^3)-36x^4\sin(2x^3)$ and $f'''(x)=12\cos(2x^3)-216x^3\sin(2x^3)+216x^6\cos(2x^3)$.

In particular, $f(0)=0$, $f'(0)=0$, $f''(0)=0$ and $f'''(0)=12$.

It follows that $f(x)=f(0)+f'(0)x+\frac{1}{2}f''(0)x^2+\ldots=0+0x+0x^2+\frac{12}{3!}x^3+\ldots=2x^3+\ldots$

Solution. (via series for sine) As done in the previous example for $\cos(x)$, we can derive that

$$\sin(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}=x-\frac{1}{6}x^3+\frac{1}{120}x^5-\ldots$$

It follows that

$$\begin{aligned}\sin(2x^3) &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(2x^3)^{2n+1}=\sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n+1}}{(2n+1)!}x^{6n+3}\\ &= \frac{2^1}{1!}x^3-\frac{2^3}{3!}x^9+\frac{2^5}{5!}x^{15}-\ldots=2x^3-\frac{4}{3}x^9+\frac{4}{15}x^{15}-\ldots\end{aligned}$$

Power series solutions to DE

Given any DE, we can approximate analytic solutions by working with the first few terms of the power series.

Example 109. (Airy equation, part I) Let $y(x)$ be the unique solution to the IVP $y''=xy$, $y(0)=a$, $y'(0)=b$. Determine the first several terms (up to x^6) in the power series of $y(x)$.

Solution. (successive differentiation) From the DE, $y''(0)=0\cdot y(0)=0$.

Differentiating both sides of the DE, we obtain $y'''=y+xy'$ so that $y'''(0)=y(0)+0\cdot y'(0)=a$.

Likewise, $y^{(4)}=2y'+xy''$ shows $y^{(4)}(0)=2y'(0)=2b$.

Continuing, $y^{(5)}=3y''+xy'''$ so that $y^{(5)}(0)=3y''(0)=0$.

Continuing, $y^{(6)}=4y'''+xy^{(4)}$ so that $y^{(6)}(0)=4y'''(0)=4a$.

Hence, $y(x)=a+bx+\frac{1}{2}y''(0)x^2+\frac{1}{6}y'''(0)x^3+\frac{1}{24}y^{(4)}(0)x^4+\frac{1}{120}y^{(5)}(0)x^5+\frac{1}{720}y^{(6)}(0)x^6+\ldots$
$=a+bx+\frac{a}{6}x^3+\frac{b}{12}x^4+\frac{a}{180}x^6+\ldots$

Comment. Do you see the general pattern? We will revisit this example soon.

Solution. (plug in power series) The powers series $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$ becomes $y = a + b x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$ because of the initial conditions.

To determine a_2, a_3, a_4, a_5, a_6, we equate the coefficients of:

$$\begin{aligned} y'' &= 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + 30a_6 x^4 + \ldots \\ x y &= a x + b x^2 + a_2 x^3 + a_3 x^4 + \ldots \end{aligned}$$

We find $2a_2 = 0$, $6a_3 = a$, $12a_4 = b$, $20a_5 = a_2$, $30a_6 = a_3$.

So $a_2 = 0$, $a_3 = \frac{a}{6}$, $a_4 = \frac{b}{12}$, $a_5 = \frac{a_2}{20} = 0$, $a_6 = \frac{a_3}{30} = \frac{a}{180}$. Hence, $y(x) = a + b x + \frac{a}{6} x^3 + \frac{b}{12} x^4 + \frac{a}{180} x^6 + \ldots$