In the special case that $\Phi(t) = e^{At}$, some things become easier. For instance, $\Phi(t)^{-1} = e^{-At}$. In that case, we can explicitely write down solutions to IVPs:

- $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{c}$ has (unique) solution $\boldsymbol{y}(t) = e^{At}\boldsymbol{c}$.
- $\boldsymbol{y}' = A\boldsymbol{y} + \boldsymbol{f}(t)$, $\boldsymbol{y}(0) = \boldsymbol{c}$ has (unique) solution $\boldsymbol{y}(t) = e^{At}\boldsymbol{c} + e^{At}\int_0^t e^{-As}\boldsymbol{f}(s)\mathrm{d}s$.

**Example 98.** Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$.

(a) Determine $e^{At}$.

(b) Solve $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(c) Solve $\boldsymbol{y}' = A\boldsymbol{y} + \begin{bmatrix} 0 \\ 2e^t \end{bmatrix}$, $\boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

**Solution.**

(a) By proceeding as in Example 73 (do it!), we find $e^{At} = \begin{bmatrix} 2e^{2t} - e^{3t} & -2e^{2t} + 2e^{3t} \\ e^{2t} - e^{3t} & -e^{2t} + 2e^{3t} \end{bmatrix}$.

(b) $\boldsymbol{y}(t) = e^{At}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2e^{2t} + 3e^{3t} \\ -e^{2t} + 3e^{3t} \end{bmatrix}$

(c) $\boldsymbol{y}(t) = e^{At}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + e^{At}\int_0^t e^{-As}\boldsymbol{f}(s)\mathrm{d}s$. We compute:

$\int_0^t e^{-As}\boldsymbol{f}(s)\mathrm{d}s = \int_0^t \begin{bmatrix} 2e^{-2s} - e^{-3s} & -2e^{-2s} + 2e^{-3s} \\ e^{-2s} - e^{-3s} & -e^{-2s} + 2e^{-3s} \end{bmatrix}\begin{bmatrix} 0 \\ 2e^s \end{bmatrix}\mathrm{d}s = \int_0^t \begin{bmatrix} -4e^{-s} + 4e^{-2s} \\ -2e^{-s} + 4e^{-2s} \end{bmatrix}\mathrm{d}s = \begin{bmatrix} 4e^{-t} - 2e^{-2t} - 2 \\ 2e^{-t} - 2e^{-2t} \end{bmatrix}$

Hence, $e^{At}\int_0^t e^{-As}\boldsymbol{f}(s)\mathrm{d}s = \begin{bmatrix} 2e^{2t} - e^{3t} & -2e^{2t} + 2e^{3t} \\ e^{2t} - e^{3t} & -e^{2t} + 2e^{3t} \end{bmatrix}\begin{bmatrix} 4e^{-t} - 2e^{-2t} - 2 \\ 2e^{-t} - 2e^{-2t} \end{bmatrix} = \begin{bmatrix} 2e^t - 4e^{2t} + 2e^{3t} \\ -2e^{2t} + 2e^{3t} \end{bmatrix}$.

Finally, $\boldsymbol{y}(t) = \begin{bmatrix} -2e^{2t} + 3e^{3t} \\ -e^{2t} + 3e^{3t} \end{bmatrix} + \begin{bmatrix} 2e^t - 4e^{2t} + 2e^{3t} \\ -2e^{2t} + 2e^{3t} \end{bmatrix} = \begin{bmatrix} 2e^t - 6e^{2t} + 5e^{3t} \\ -3e^{2t} + 5e^{3t} \end{bmatrix}$.

**Sage.** Here is how we can let Sage do these computations for us:

```
>>> s, t = var('s, t')
>>> A = matrix([[1,2],[-1,4]])
>>> y = exp(A*t)*vector([1,2]) + exp(A*t)*integrate(exp(-A*s)*vector([0,2*e^s]), s,0,t)
>>> y.simplify_full()
```

$(5\, e^{(3\,t)} - 6\, e^{(2\,t)} + 2\, e^t,\, 5\, e^{(3\,t)} - 3\, e^{(2\,t)})$

**Comment.** Can you see that the solution is of the form that we anticipate from the method of undetermined coefficients?
Indeed, $\boldsymbol{y}(t) = \boldsymbol{y}_p(t) + \boldsymbol{y}_h(t)$ where the simplest particular solution is $\boldsymbol{y}_p(t) = \begin{bmatrix} 2e^t \\ 0 \end{bmatrix}$.

# Modeling & Applications

## Mixing problems

**Example 99.** Consider two brine tanks. Tank $T_1$ contains 24gal water containing 3lb salt, and tank $T_2$ contains 9gal pure water.

- $T_1$ is being filled with 54gal/min water containing 0.5lb/gal salt.
- 72gal/min well-mixed solution flows out of $T_1$ into $T_2$.
- 18gal/min well-mixed solution flows out of $T_2$ into $T_1$.
- Finally, 54gal/min well-mixed solution is leaving $T_2$.

We wish to understand how much salt is in the tanks after $t$ minutes.

(a) Derive a system of differential equations.

(b) Determine the equilibrium points and classify their stability. What does this mean here?

(c) Solve the system to find explicit formulas for how much salt is in the tanks after $t$ minutes.

**Solution.**

(a) Note that the amount of water in each tank is constant because the flows balance each other.
Let $y_i(t)$ denote the amount of salt (in lb) in tank $T_i$ after time $t$ (in min). In time interval $[t, t+\Delta t]$:
$\Delta y_1 \approx 54 \cdot \frac{1}{2} \cdot \Delta t - 72 \cdot \frac{y_1}{24} \cdot \Delta t + 18 \cdot \frac{y_2}{9} \cdot \Delta t$, so $y_1' = 27 - 3y_1 + 2y_2$. Also, $y_1(0) = 3$.
$\Delta y_2 \approx 72 \cdot \frac{y_1}{24} \cdot \Delta t - 72 \cdot \frac{y_2}{9} \cdot \Delta t$, so $y_2' = 3y_1 - 8y_2$. Also, $y_2(0) = 0$.
Using matrix notation and writing $\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, this is $\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{y} = \begin{bmatrix} -3 & 2 \\ 3 & -8 \end{bmatrix}\boldsymbol{y} + \begin{bmatrix} 27 \\ 0 \end{bmatrix}$, $\boldsymbol{y}(0) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$.

(b) Note that this system is autonomous! Otherwise, we could not pursue our stability analysis.
To find the equilibrium point (since the system is linear, there should be just one), we set $\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{y} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and solve $\begin{bmatrix} -3 & 2 \\ 3 & -8 \end{bmatrix}\boldsymbol{y} + \begin{bmatrix} 27 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. We find $\boldsymbol{y} = \begin{bmatrix} -3 & 2 \\ 3 & -8 \end{bmatrix}^{-1}\begin{bmatrix} -27 \\ 0 \end{bmatrix} = \begin{bmatrix} 12 \\ 4.5 \end{bmatrix}$.
The characteristic polynomial of $\begin{bmatrix} -3 & 2 \\ 3 & -8 \end{bmatrix}$ is $(-3-\lambda)(-8-\lambda) - 6 = \lambda^2 + 11\lambda + 18 = (\lambda+9)(\lambda+2)$.
Hence, the eigenvalues are $-9, -2$. Since they are both negative, the equilibrium point is a nodal sink and, in particular, asymptotically stable.
Having an equilibrium point at $(12, 4.5)$, means that, if the salt amounts are $y_1 = 12$, $y_2 = 4.5$, then they won't change over time (but will remain unchanged at these levels). The fact that it is asymptotically stable means that salt amounts close to these balanced levels will, over time, approach the equilibrium levels. (Because the system is linear, this is also true for levels that are not "close".)
**We could have "seen" the equilibrium point!**
Indeed, noticing that, for a constant (equilibrium) particular solution $\boldsymbol{y}$, each tank has to have a constant concentration of 0.5lb/gal of salt, we find directly $\boldsymbol{y} = 0.5\begin{bmatrix} 24 \\ 9 \end{bmatrix} = \begin{bmatrix} 12 \\ 4.5 \end{bmatrix}$.

(c) This is an IVP that we can solve (with some work). Do it! For instance, we can apply variation of constants. (Alternatively, leverage our particular solution from the previous part!) Skipping most work, we find:

- If $A = \begin{bmatrix} -3 & 2 \\ 3 & -8 \end{bmatrix}$, then $e^{At} = \frac{1}{7}\begin{bmatrix} e^{-9t} + 6e^{-2t} & -2e^{-9t} + 2e^{-2t} \\ -3e^{-9t} + 3e^{-2t} & 6e^{-9t} + 1e^{-2t} \end{bmatrix}$
- $\boldsymbol{y} = e^{At}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{At}\int_0^t e^{-As}\begin{bmatrix} 27 \\ 0 \end{bmatrix}\mathrm{d}s = \frac{1}{7}\begin{bmatrix} e^{-9t} + 6e^{-2t} \\ -3e^{-9t} + 3e^{-2t} \end{bmatrix} + \frac{3}{14}e^{At}\begin{bmatrix} 2e^{9t} + 54e^{2t} - 56 \\ -6e^{9t} + 27e^{2t} - 21 \end{bmatrix}$
  $= \begin{bmatrix} 12 - 9e^{-2t} \\ 4.5 - 4.5e^{-2t} \end{bmatrix}$