

Review. The matrix exponential and how to compute it.

Excursion: Euler's identity

Theorem 87. (Euler's identity) $e^{ix} = \cos(x) + i \sin(x)$

Proof. Observe that both sides are the (unique) solution to the IVP $y' = iy, y(0) = 1$.
 [Check that by computing the derivatives and verifying the initial condition! As we did in class.] □

On lots of T-shirts. In particular, with $x = \pi$, we get $e^{i\pi} = -1$ or $e^{i\pi} + 1 = 0$ (which connects the five fundamental constants).

Example 88. Where do trig identities like $\sin(2x) = 2\cos(x)\sin(x)$ or $\sin^2(x) = \frac{1 - \cos(2x)}{2}$ (and infinitely many others you have never heard of!) come from?

Short answer: they all come from the simple exponential law $e^{x+y} = e^x e^y$.

Let us illustrate this in the simple case $(e^x)^2 = e^{2x}$. Observe that

$$\begin{aligned} e^{2ix} &= \cos(2x) + i \sin(2x) \\ e^{ix} e^{ix} &= [\cos(x) + i \sin(x)]^2 = \cos^2(x) - \sin^2(x) + 2i \cos(x)\sin(x). \end{aligned}$$

Comparing imaginary parts (the "stuff with an i "), we conclude that $\sin(2x) = 2\cos(x)\sin(x)$.
 Likewise, comparing real parts, we read off $\cos(2x) = \cos^2(x) - \sin^2(x)$.

(Use $\cos^2(x) + \sin^2(x) = 1$ to derive $\sin^2(x) = \frac{1 - \cos(2x)}{2}$ from the last equation.)

Challenge. Can you find a triple-angle trig identity for $\cos(3x)$ and $\sin(3x)$ using $(e^x)^3 = e^{3x}$?

Or, use $e^{i(x+y)} = e^{ix} e^{iy}$ to derive $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ and $\sin(x+y) = \dots$ (that's what we actually did in class).

Realize that the complex number $e^{i\theta} = \cos(\theta) + i \sin(\theta)$ corresponds to the point $(\cos(\theta), \sin(\theta))$. These are precisely the points on the unit circle!

Recall that a point (x, y) can be represented using **polar coordinates** (r, θ) , where r is the distance to the origin and θ is the angle with the x -axis.

Then, $x = r \cos\theta$ and $y = r \sin\theta$.

Every complex number z can be written in **polar form** as $z = r e^{i\theta}$, with $r = |z|$.

Why? By comparing with the usual polar coordinates $(x = r \cos\theta$ and $y = r \sin\theta)$, we can write

$$z = x + iy = r \cos\theta + ir \sin\theta = r e^{i\theta}.$$

In the final step, we used Euler's identity.