Example 70. (review) Write the (third-order) differential equation $y''' = 3y'' - 2y' + y$ as a system of (first-order) differential equations.

Solution. If $\boldsymbol{y} = \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix}$, then $\boldsymbol{y}' = \begin{bmatrix} y' \\ y'' \\ y''' \end{bmatrix} = \begin{bmatrix} y' \\ y'' \\ 3y'' - 2y' + y \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -2 & 3 \end{bmatrix} \boldsymbol{y}$.

Example 71. Consider the following system of (second-order) initial value problems:

$$\begin{aligned} y_1'' &= 2y_1' - 3y_2' + 7y_2 \\ y_2'' &= 4y_1' + y_2' - 5y_1 \end{aligned} \qquad y_1(0) = 2,\ y_1'(0) = 3,\ y_2(0) = -1,\ y_2'(0) = 1$$

Write it as a first-order initial value problem in the form $\boldsymbol{y}' = M\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$.

Solution. If $\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_1' \\ y_2' \end{bmatrix}$, then $\boldsymbol{y}' = \begin{bmatrix} y_1' \\ y_2' \\ y_1'' \\ y_2'' \end{bmatrix} = \begin{bmatrix} y_1' \\ y_2' \\ 2y_1' - 3y_2' + 7y_2 \\ 4y_1' + y_2' - 5y_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 7 & 2 & -3 \\ -5 & 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 7 & 2 & -3 \\ -5 & 0 & 4 & 1 \end{bmatrix} \boldsymbol{y}$.

For short, the system translates into $\boldsymbol{y}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 7 & 2 & -3 \\ -5 & 0 & 4 & 1 \end{bmatrix} \boldsymbol{y}$ with $\boldsymbol{y}(0) = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \end{bmatrix}$.

Example 72. Suppose that $e^{Mx} = \frac{1}{10} \begin{bmatrix} e^x + 9e^{2x} & 3e^x - 3e^{2x} \\ 3e^x - 3e^{2x} & 9e^x + e^{2x} \end{bmatrix}$.

(a) Without doing any computations, determine M^n.

(b) What is M?

(c) Without doing any computations, determine the eigenvalues and eigenvectors of M.

(d) From these eigenvalues and eigenvectors, write down a simple fundamental matrix solution to $\boldsymbol{y}' = M\boldsymbol{y}$.

(e) From that fundamental matrix solution, how can we compute e^{Mx}? (If we didn't know it already...)

(f) Having computed e^{Mx}, what is a simple check that we can (should!) make?

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Solution.

(a) Since e^x and e^{2x} correspond to eigenvalues 1 and 2, we just need to replace these by $1^n = 1$ and 2^n:

$$M^n = \frac{1}{10}\begin{bmatrix} 1+9\cdot 2^n & 3-3\cdot 2^n \\ 3-3\cdot 2^n & 9+2^n \end{bmatrix}$$

(b) We can simply set $n=1$ in our formula for M^n, to get $M = \frac{1}{10}\begin{bmatrix} 19 & -3 \\ -3 & 11 \end{bmatrix}$.

(c) The eigenvalues are 1 and 2 (because e^{Mx} contains the exponentials e^x and e^{2x}).

Looking at the coefficients of e^x in the first column of e^{Mx}, we see that $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is a 1-eigenvector.

[We can also look the second column of e^{Mx}, to obtain $\begin{bmatrix} 3 \\ 9 \end{bmatrix}$ which is a multiple and thus equivalent.]

Likewise, by looking at the coefficients of e^{2x}, we see that $\begin{bmatrix} 9 \\ -3 \end{bmatrix}$ or, equivalently, $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$ is a 2-eigenvector.

Comment. To see where this is coming from, keep in mind that, associated to a λ-eigenvector $\boldsymbol{v}$, we have the corresponding solution $\boldsymbol{y}(x) = \boldsymbol{v}e^{\lambda x}$ of the DE $\boldsymbol{y}' = M\boldsymbol{y}$. On the other hand, the columns of e^{Mx} are solutions to that DE and, therefore, must be linear combinations of these $\boldsymbol{v}e^{\lambda x}$.

(d) From the eigenvalues and eigenvectors, we know that $\begin{bmatrix} 1 \\ 3 \end{bmatrix}e^x$ and $\begin{bmatrix} -3 \\ 1 \end{bmatrix}e^{2x}$ are solutions (and that the general solutions consists of the linear combinations of these two).

Selecting these as the columns, we obtain the fundamental matrix solution $\Phi(x) = \begin{bmatrix} e^x & -3e^{2x} \\ 3e^x & e^{2x} \end{bmatrix}$.

Comment. The *fundamental* refers to the fact that the columns combine to the general solution.

The *matrix solution* means that $\Phi(x)$ itself satisfies the DE: namely, we have $\Phi' = M\Phi$. That this is the case is a consequence of matrix multiplication (namely, say, the second column of $M\Phi$ is defined to be M times the second column of Φ; but that column is a vector solution and therefore solves the DE).

(e) We can compute e^{Mx} as $e^{Mx} = \Phi(x)\Phi(0)^{-1}$.

If $\Phi(x) = \begin{bmatrix} e^x & -3e^{2x} \\ 3e^x & e^{2x} \end{bmatrix}$, then $\Phi(0) = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}$ and, hence, $\Phi(0)^{-1} = \frac{1}{10}\begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}$. It follows that

$$e^{Mx} = \Phi(x)\Phi(0)^{-1} = \begin{bmatrix} e^x & -3e^{2x} \\ 3e^x & e^{2x} \end{bmatrix}\frac{1}{10}\begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} e^x + 9e^{2x} & 3e^x - 3e^{2x} \\ 3e^x - 3e^{2x} & 9e^x + e^{2x} \end{bmatrix}.$$

(f) We can check that e^{Mx} equals the identity matrix if we set $x=0$:

$$\frac{1}{10}\begin{bmatrix} e^x + 9e^{2x} & 3e^x - 3e^{2x} \\ 3e^x - 3e^{2x} & 9e^x + e^{2x} \end{bmatrix} \overset{x=0}{\rightsquigarrow} \frac{1}{10}\begin{bmatrix} 1+9 & 3-3 \\ 3-3 & 9+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This check does not require much effort and can even be done in our head while writing down e^{Mx}. There is really no excuse for not doing it!

Example 73. (homework) Let $M = \begin{bmatrix} -1 & 6 \\ -1 & 4 \end{bmatrix}$.

(a) Determine the general solution to $\boldsymbol{y}' = M\boldsymbol{y}$.

(b) Determine a fundamental matrix solution to $\boldsymbol{y}' = M\boldsymbol{y}$.

(c) Compute e^{Mx}.

(d) Solve the initial value problem $\boldsymbol{y}' = M\boldsymbol{y}$ with $\boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(e) Compute M^n.

(f) Solve $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$ with $\boldsymbol{a}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Solution.

(a) We determine the eigenvectors of M. The characteristic polynomial is:

$\det(M - \lambda I) = \det\left(\begin{bmatrix} -1-\lambda & 6 \\ -1 & 4-\lambda \end{bmatrix}\right) = (-1-\lambda)(4-\lambda) + 6 = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2)$

Hence, the eigenvalues are $\lambda = 1$ and $\lambda = 2$.

- $\lambda = 1$: Solving $\begin{bmatrix} -2 & 6 \\ -1 & 3 \end{bmatrix}\boldsymbol{v} = \boldsymbol{0}$, we find that $\boldsymbol{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 1$.
- $\lambda = 2$: Solving $\begin{bmatrix} -3 & 6 \\ -1 & 2 \end{bmatrix}\boldsymbol{v} = \boldsymbol{0}$, we find that $\boldsymbol{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 2$.

Hence, the general solution is $C_1\begin{bmatrix} 3 \\ 1 \end{bmatrix}e^x + C_2\begin{bmatrix} 2 \\ 1 \end{bmatrix}e^{2x}$.

(b) The corresponding fundamental matrix solution is $\Phi = \begin{bmatrix} 3e^x & 2e^{2x} \\ e^x & e^{2x} \end{bmatrix}$.

(c) Note that $\Phi(0) = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, so that $\Phi(0)^{-1} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$. It follows that

$$e^{Mx} = \Phi(x)\Phi(0)^{-1} = \begin{bmatrix} 3e^x & 2e^{2x} \\ e^x & e^{2x} \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 3e^x - 2e^{2x} & -6e^x + 6e^{2x} \\ e^x - e^{2x} & -2e^x + 3e^{2x} \end{bmatrix}.$$

(d) The solution to the IVP is $\boldsymbol{y}(x) = e^{Mx}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3e^x - 2e^{2x} & -6e^x + 6e^{2x} \\ e^x - e^{2x} & -2e^x + 3e^{2x} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3e^x + 4e^{2x} \\ -e^x + 2e^{2x} \end{bmatrix}$.

Note. If we hadn't already computed e^{Mx}, we would use the general solution and solve for the appropriate values of C_1 and C_2. Do it that way as well!

(e) From the first part, it follows that $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$ has general solution $C_1\begin{bmatrix} 3 \\ 1 \end{bmatrix} + C_2\begin{bmatrix} 2 \\ 1 \end{bmatrix}2^n$.

(Note that $1^n = 1$.)

The corresponding fundamental matrix solution is $\Phi_n = \begin{bmatrix} 3 & 2 \cdot 2^n \\ 1 & 2^n \end{bmatrix}$.

As above, $\Phi_0 = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, so that $\Phi(0)^{-1} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$ and

$$M^n = \Phi_n\Phi_0^{-1} = \begin{bmatrix} 3 & 2 \cdot 2^n \\ 1 & 2^n \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 3 - 2 \cdot 2^n & -6 + 6 \cdot 2^n \\ 1 - 2^n & -2 + 3 \cdot 2^n \end{bmatrix}.$$

Important. Compare with our computation for e^{Mx}. Can you see how this was basically the same computation? Write down M^n directly from e^{Mx}.

(f) The (unique) solution is $\boldsymbol{a}_n = M^n\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 - 2 \cdot 2^n & -6 + 6 \cdot 2^n \\ 1 - 2^n & -2 + 3 \cdot 2^n \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 + 4 \cdot 2^n \\ -1 + 2 \cdot 2^n \end{bmatrix}$.

Important. Again, compare with the earlier IVP! Without work, we can write down one from the other.