Example 34. (review) What is the shape of a particular solution of $y''+4y'+4y=4e^{3x}\sin(2x)-x\sin(x)$?

Solution. The "old" roots are $-2,-2$. The "new" roots are $3\pm 2i, \pm i, \pm i$.

Hence, there has to be a particular solution of the form

$y_p = C_1 e^{3x}\cos(2x) + C_2 e^{3x}\sin(2x) + (C_3 + C_4 x)\cos(x) + (C_5 + C_6 x)\sin(x)$.

Continuing to find a particular solution. To find the values of $C_1, \ldots, C_6$, we plug into the DE. But this final step is so boring that we don't go through it here. Computers (currently?) cannot afford to be as selective; mine obediently calculated: $y_p = -\frac{4}{841}e^{3x}(20\cos(2x) - 21\sin(2x)) + \frac{1}{125}((-22+20x)\cos(x) + (4-15x)\sin(x))$

Sage

In practice, we are happy to let a machine do tedious computations. Let us see how to use the open-source computer algebra system **Sage** to do basic computations for us.

Sage is freely available at `sagemath.org`. Instead of installing it locally (it's huge!) we can conveniently use it in the cloud at `cocalc.com` from any browser.

[For basic computations, you can also simply use the textbox on our course website.]

Sage is built as a **Python** library, so any Python code is valid. For starters, we will use it as a fancy calculator.

Example 35. To solve the differential equation $y''+4y'+4y=7e^{-2x}$, as we did in Example 30, we can use the following:

```
>>> x = var('x')

>>> y = function('y')(x)

>>> desolve(diff(y,x,2) + 4*diff(y,x) + 4*y == 7*exp(-2*x), y)
```

$$\frac{7}{2}x^2 e^{(-2x)} + (K_2 x + K_1)e^{(-2x)}$$

This confirms, as we had found, that the general solution is $y(x) = \left(C_1 + C_2 x + \frac{7}{2}x^2\right)e^{-2x}$.

Example 36. Similarly, Sage can solve initial value problems such as $y''-y'-2y=0$ with initial conditions $y(0)=4$, $y'(0)=5$.

```
>>> x = var('x')

>>> y = function('y')(x)

>>> desolve(diff(y,x,2) - diff(y,x) - 2*y == 0, y, ics=[0,4,5])
```

$$3e^{(2x)} + e^{(-x)}$$

This matches the (unique) solution $y(x) = 3e^{2x} + e^{-x}$ that we derived in Example 18.

Armin Straub
straub@southalabama.edu

Example 37. We have been factoring differential operators like $D^2 + 4D + 4 = (D+2)^2$.

Things become much more complicated when the coefficients are not constant!

For instance, the linear DE $y'' + 4y' + 4xy = 0$ can be written as $Ly = 0$ with $L = D^2 + 4D + 4x$. However, in general, such operators cannot be factored (unless we allow as coefficients functions in x that we are not familiar with). [On the other hand, any ordinary polynomial can be factored over the complex numbers.]

One indication that things become much more complicated is that x and D do not commute: $xD \neq Dx$!!

Indeed, $(xD)f(x) = xf'(x)$ while $(Dx)f(x) = \frac{\mathrm{d}}{\mathrm{d}x}[xf(x)] = f(x) + xf'(x) = (1 + xD)f(x)$.

This computation shows that, in fact, $Dx = xD + 1$.

Review. Linear DEs are those that can be written as $Ly = f(x)$ where L is a linear differential operator: namely,

$$L = p_n(x)D^n + p_{n-1}(x)D^{n-1} + \ldots + p_1(x)D + p_0(x). \tag{1}$$

Recall that the operators xD and Dx are not the same: instead, $Dx = xD + 1$.

We say that an operator of the form (1) is in **normal form**.

For instance. xD is in normal form, whereas Dx is not in normal form. It follows from the previous example that the normal form of Dx is $xD + 1$.

Example 38. Let $a = a(x)$ be some function.

(a) Write the operator Da in normal form [normal form means as in (1)].

(b) Write the operator D^2a in normal form.

Solution.

(a) $(Da)f(x) = \frac{\mathrm{d}}{\mathrm{d}x}[a(x)\,f(x)] = a'(x)f(x) + a(x)f'(x) = (a' + aD)f(x)$

Hence, $Da = aD + a'$.

(b) $(D^2a)f(x) = \frac{\mathrm{d}^2}{\mathrm{d}x^2}[a(x)\,f(x)] = \frac{\mathrm{d}}{\mathrm{d}x}[a'(x)f(x) + a(x)f'(x)] = a''(x)f(x) + 2a'(x)f'(x) + a(x)f''(x)$
$= (a'' + 2a'D + aD^2)f(x)$

Hence, $D^2a = aD^2 + 2a'D + a''$.