Midterm #2

MATH 332 — Differential Equations II
Wednesday, Nov 15, 2023

Please print your name:

No notes, calculators or tools of any kind are permitted. There are 31 points in total. You need to show work to receive full credit.

Good luck!

Problem 1. (7 points) Determine the equilibrium points of the system $\frac{dx}{dt} = (x-2)y$, $\frac{dy}{dt} = xy - 1$ and classify their stability.

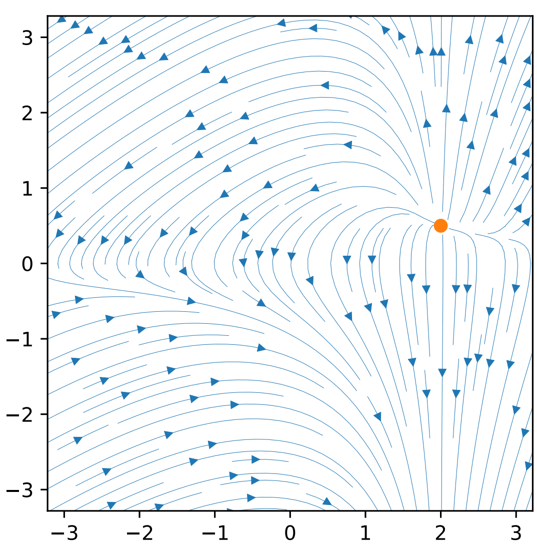
Solution. To find the equilibrium points, we solve $(x-2)y = 0$ and $xy - 1 = 0$. The first equation implies that we have $x = 2$ or $y = 0$. If $x = 2$, then the second equation implies $y = \frac{1}{2}$. On the other hand, if $y = 0$, then the second equation has no solution. We conclude that the only equilibrium point is $\left(2, \frac{1}{2}\right)$.

Our system is $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$ with $\begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} = \begin{bmatrix} (x-2)y \\ xy-1 \end{bmatrix}$.

The Jacobian matrix is $J(x,y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} y & x-2 \\ y & x \end{bmatrix}$.

At $\left(2, \frac{1}{2}\right)$, the Jacobian matrix is $J\left(2, \frac{1}{2}\right) = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 2 \end{bmatrix}$. We can read off that the eigenvalues are $\frac{1}{2}, 2$. Since both are positive, $\left(2, \frac{1}{2}\right)$ is a nodal source. In particular, $\left(2, \frac{1}{2}\right)$ is unstable.

The following phase portrait confirms our analysis:



Problem 2. (3 points) A mass-spring system is described by the equation $my'' + 2y = \sum_{n=1}^{\infty} \frac{1}{2n^2} \cos\left(\frac{nt}{3}\right)$.

For which values of m does resonance occur?

Solution. The roots of $p(D) = mD^2 + 2$ are $\pm i\sqrt{\frac{2}{m}}$, so that that the natural frequency is $\sqrt{\frac{2}{m}}$. Resonance therefore occurs if $\sqrt{\frac{2}{m}} = \frac{n}{3}$ for some $n \in \{1, 2, 3, \ldots\}$. Equivalently, resonance occurs if $m = \frac{18}{n^2}$ for some $n \in \{1, 2, 3, \ldots\}$.

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Problem 3. (3 points) Let $y(x)$ be the unique solution to the IVP $y''=1+2(x-1)y^2$, $y(0)=1$, $y'(0)=2$. Determine the first several terms (up to x^3) in the power series of $y(x)$.

Solution. (successive differentiation) From the DE, $y''(0)=1+2\cdot(-1)\cdot y(0)^2=-1$.

Differentiating both sides of the DE, we obtain $y'''=2y^2+4(x-1)\,yy'$. In particular, $y'''(0)=2y(0)^2+4\cdot(-1)\cdot y(0)\cdot y'(0)=-6$.

Hence, $y(x)=y(0)+y'(0)x+\frac{1}{2}y''(0)x^2+\frac{1}{6}y'''(0)x^3+\ldots=1+2x-\frac{1}{2}x^2-x^3+\ldots$

Solution. (plug in power series) Taking into account the initial conditions, $y=1+2x+a_2x^2+a_3x^3+\ldots$.

Therefore, $y''=2a_2+6a_3x+\ldots$

On the other hand, $y^2=1+4x+\ldots$ so that $1+2(x-1)y^2=1+2(x-1)(1+4x+\ldots)=-1-6x+\ldots$

Equating coefficients of y'' and $1+2(x-1)y^2$, we find $2a_2=-1$ and $6a_3=-6$.

So $a_2=-\frac{1}{2}$, $a_3=-1$ and, hence, $y(x)=1+2x-\frac{1}{2}x^2-x^3+\ldots$

Problem 4. (3 points) Find a minimum value for the radius of convergence of a power series solution to

$$(x-3)y''=\frac{2y+1}{x^2+1} \quad \text{at } x=1.$$

Solution. Note that this is a linear DE! (Otherwise, we could not proceed.) Rewriting the DE as $y''-\frac{2}{(x^2+1)(x-3)}y=\frac{1}{(x^2+1)(x-3)}$, we see that the singular points are $x=\pm i,3$.

Note that $x=1$ is an ordinary point of the DE and that the distance to the nearest singular point is $|1-(\pm i)|=\sqrt{1^2+1^2}=\sqrt{2}$ (the distance to 3 is $|1-3|=2>\sqrt{2}$).

Hence, the DE has power series solutions about $x=1$ with radius of convergence at least $\sqrt{2}$.

Problem 5. (6 points) Derive a recursive description of a power series solution $y(x)$ (around $x=0$) to the differential equation $y''=x^2y'+3y$.

Solution. Let us spell out the power series for y, x^2y', y'':

$$y(x)=\sum_{n=0}^{\infty} a_nx^n$$

$$x^2y'(x)=\sum_{n=1}^{\infty} n a_nx^{n+1}=\sum_{n=2}^{\infty}(n-1)a_{n-1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty} n(n-1)a_nx^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

Hence, the DE becomes:

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n=\sum_{n=2}^{\infty}(n-1)a_{n-1}x^n+3\sum_{n=0}^{\infty}a_nx^n.$$

We compare coefficients of x^n:

- $n=0$: $2a_2=3a_0$, so that $a_2=\frac{3}{2}a_0$.
- $n=1$: $6a_3=3a_1$, so that $a_3=\frac{1}{2}a_1$.
- $n\geqslant 2$: $(n+2)(n+1)a_{n+2}=(n-1)a_{n-1}+3a_n$

  Equivalently, for $n\geqslant 4$, $a_n=\frac{3}{n(n-1)}a_{n-2}+\frac{n-3}{n(n-1)}a_{n-3}$. (Can you see why this also holds for $n=3$?)

In conclusion, the power series $y(x)=\sum_{n=0}^{\infty}a_nx^n$ is recursively determined by

$$a_2=\frac{3}{2}a_0, \quad a_n=\frac{3}{n(n-1)}a_{n-2}+\frac{n-3}{n(n-1)}a_{n-3} \quad \text{for } n\geqslant 3.$$

(The values a_0 and a_1 are the initial conditions.)

Problem 6. (5 points)

(a) Suppose $y(x) = \sum_{n=0}^{\infty} a_n (x+2)^n$. How can we compute the a_n from $y(x)$? $a_n = \boxed{}$.

(b) Suppose $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(3n\pi t) + b_n \sin(3n\pi t))$. How can we compute the a_n and b_n from $f(t)$?

$a_n = \boxed{}$ and $b_n = \boxed{}$.

(c) Determine the power series around $x = 0$: $\frac{3}{1+7x} = \boxed{}$

(d) Determine the power series around $x = 0$: $e^{-3x} = \boxed{}$

Solution.

(a) $a_n = \frac{y^{(n)}(-2)}{n!}$ because this is the Taylor series of $f(x)$ around $x = -2$.

(b) This is the Fourier series of $f(t)$ (which has period $\frac{2}{3}$ and so is $2L$-periodic with $L = \frac{1}{3}$). The Fourier coefficients a_n, b_n can be computed as

$$a_n = 3\int_{-\frac{1}{3}}^{\frac{1}{3}} f(t)\cos(3n\pi t)\mathrm{d}t, \qquad b_n = 3\int_{-\frac{1}{3}}^{\frac{1}{3}} f(t)\sin(3n\pi t)\mathrm{d}t.$$

(c) Since $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, we have $\frac{3}{1+7x} = 3\sum_{n=0}^{\infty} (-7x)^n = 3\sum_{n=0}^{\infty} (-7)^n x^n$.

(d) Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, we have $e^{-3x} = \sum_{n=0}^{\infty} \frac{(-3)^n x^n}{n!}$.

Problem 7. (4 points) Consider the function $f(t) = 1 - t$, defined for $t \in [0, 1]$.

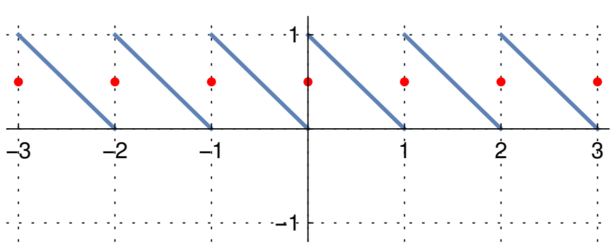
(a) Sketch the Fourier series of $f(t)$ for $t \in [-3, 3]$.

(b) Sketch the Fourier cosine series of $f(t)$ for $t \in [-3, 3]$.

(c) Sketch the Fourier sine series of $f(t)$ for $t \in [-3, 3]$.

In each sketch, carefully mark the values of the Fourier series at discontinuities.

Solution.

(a) 

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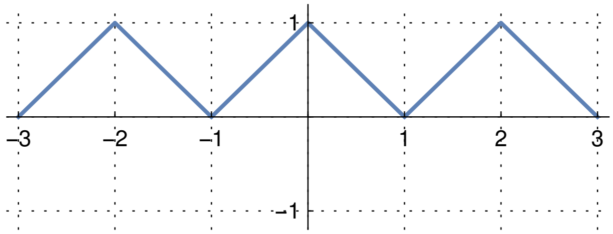
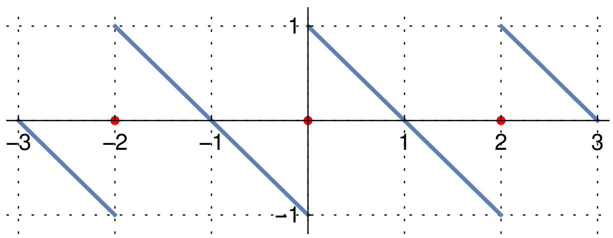
(b) 

(c) 

(extra scratch paper)