Midterm #1

MATH 332 — Differential Equations II
Wednesday, Oct 4, 2023

Please print your name:

No notes, calculators or tools of any kind are permitted. There are 30 points in total. You need to show work to receive full credit.

Good luck!

Problem 1. (10 points) Let $M = \begin{bmatrix} 1 & 3 \\ -1 & 5 \end{bmatrix}$.

(a) Compute e^{Mt}.

(b) Solve the initial value problem $\boldsymbol{y}' = M\boldsymbol{y}$ with $\boldsymbol{y}(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

(c) Determine all equilibrium points of $\begin{bmatrix} x \\ y \end{bmatrix}' = M\begin{bmatrix} x \\ y \end{bmatrix}$ and their stability.

Solution.

(a) We determine the eigenvectors of M. The characteristic polynomial is:

$\det(M - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 3 \\ -1 & 5-\lambda \end{bmatrix}\right) = (1-\lambda)(5-\lambda) + 3 = \lambda^2 - 6\lambda + 8 = (\lambda - 2)(\lambda - 4)$

Hence, the eigenvalues are $\lambda = 2$ and $\lambda = 4$.

- To find an eigenvector $\boldsymbol{v}$ for $\lambda = 2$, we need to solve $\begin{bmatrix} -1 & 3 \\ -1 & 3 \end{bmatrix}\boldsymbol{v} = \boldsymbol{0}$.

  Hence, $\boldsymbol{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 2$.

- To find an eigenvector $\boldsymbol{v}$ for $\lambda = 4$, we need to solve $\begin{bmatrix} -3 & 3 \\ -1 & 1 \end{bmatrix}\boldsymbol{v} = \boldsymbol{0}$.

  Hence, $\boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 4$.

Hence, a fundamental matrix solution is $\Phi(t) = \begin{bmatrix} 3e^{2t} & e^{4t} \\ e^{2t} & e^{4t} \end{bmatrix}$.

Note that $\Phi(0) = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$, so that $\Phi(0)^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}$. It follows that

$$e^{Mt} = \Phi(t)\Phi(0)^{-1} = \begin{bmatrix} 3e^{2t} & e^{4t} \\ e^{2t} & e^{4t} \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3e^{2t} - e^{4t} & -3e^{2t} + 3e^{4t} \\ e^{2t} - e^{4t} & -e^{2t} + 3e^{4t} \end{bmatrix}.$$

(b) The solution to the IVP is $\boldsymbol{y}(x) = e^{Mt}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3e^{2t} - e^{4t} & -3e^{2t} + 3e^{4t} \\ e^{2t} - e^{4t} & -e^{2t} + 3e^{4t} \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3e^{2t} - e^{4t} \\ e^{2t} - e^{4t} \end{bmatrix}$.

(c) The only equilibrium point is $(0, 0)$ and it is unstable.

Since M is invertible, solving $M\begin{bmatrix} x \\ y \end{bmatrix} = \boldsymbol{0}$ we only get the unique solution $\begin{bmatrix} x \\ y \end{bmatrix} = \boldsymbol{0}$, which means that only $(0, 0)$ is an equilibrium point. On the other hand, looking at e^{Mt} we see that the eigenvalues of M are 2 and 4. Because at least one eigenvalue is positive, the equilibrium point is unstable.

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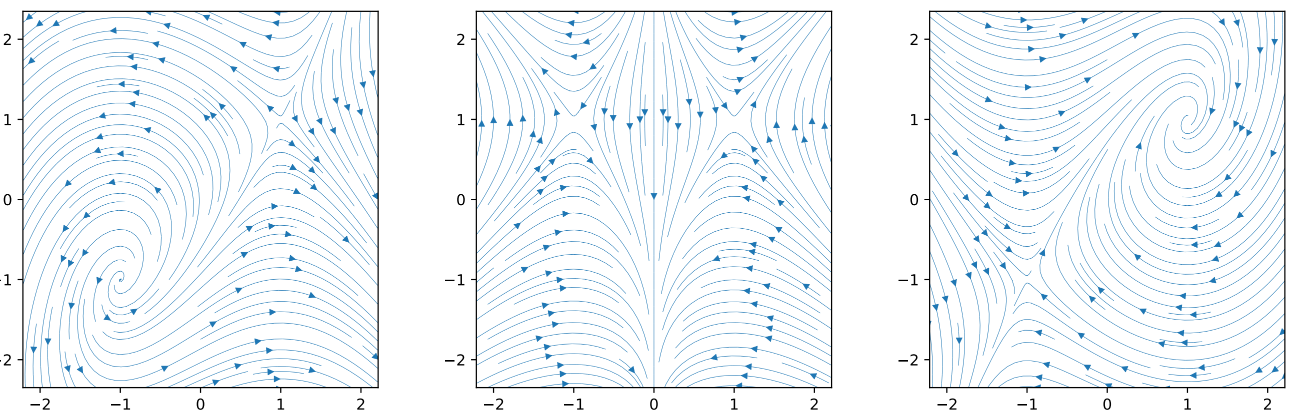
Problem 2. (3 points)

(a) Circle the phase portrait below which belongs to $\frac{\mathrm{d}x}{\mathrm{d}t} = y - x$, $\frac{\mathrm{d}y}{\mathrm{d}t} = 1 - x^2$.

(b) Determine all equilibrium points and classify the stability of each.



Solution.

(a) We can look at certain points that distinguish the plots: A good point might be $(-2, 1)$ because all three plots have a visibly different direction at that point. The differential equations tell us that $\frac{\mathrm{d}x}{\mathrm{d}t} = y - x = 3$, $\frac{\mathrm{d}y}{\mathrm{d}t} = 1 - x^2 = -3$. That means the trajectory through $(-2, 1)$ is moving in direction $\begin{bmatrix} 3 \\ -3 \end{bmatrix}$. This is only compatible with the third plot.

Thus, the third plot must be the correct one.

Comment. Computing the equilibrium points first, we can see right away that the second plot is not the correct one.

(b) We solve $y - x = 0$ (that is, $x = y$) and $1 - x^2 = 0$ (that is, $x = \pm 1$).

We conclude that the equilibrium points are $(1, 1)$ and $(-1, -1)$.

$(1, 1)$ is asymptotically stable (because all nearby solutions “flow into” it). (More precisely, this is a spiral sink.)

$(-1, -1)$ is unstable (because some nearby solutions “flow away” from it). (More precisely, this is a saddle.)

Problem 3. (8 points) Fill in the blanks. None of the problems should require any computation!

(a) Consider a homogeneous linear differential equation with constant real coefficients which has order 4. Suppose $y(x) = 7x + 2e^{-x}\cos(2x)$ is a solution. Write down the general solution.

(b) Let y_p be any solution to the inhomogeneous linear differential equation $y'' - 4y = 3e^x + 5x^2$. Find a homogeneous linear differential equation which y_p solves.

You can use the operator D to write the DE. No need to simplify, any form is acceptable.

(c) Determine a (homogeneous linear) recurrence equation satisfied by $a_n = (n+3)4^n + 5$.

You can use the operator N to write the recurrence. No need to simplify, any form is acceptable.

(d) If $e^{Mx} = \begin{bmatrix} 2e^{2x} - e^x & -2e^{2x} + 2e^x \\ e^{2x} - e^x & -e^{2x} + 2e^x \end{bmatrix}$, then $M^n =$ ______ .

Solution.

(a) $y(x) = C_1 + C_2 x + C_3 e^{-x}\cos(2x) + C_4 e^{-x}\sin(2x)$.

Explanation. $y(x) = 7x + 2e^{-x}\cos(2x)$ is a solution of $p(D)y = 0$ if and only if $0, 0, -1 \pm 2i$ are roots of the characteristic polynomial $p(D)$. Since the order of the DE is 4, there can be no further roots. Hence, the general solution of this DE is $y(x) = C_1 + C_2 x + C_3 e^{-x}\cos(2x) + C_4 e^{-x}\sin(2x)$.

(b) $(D-1)D^3(D^2-4)y = 0$

Explanation. Since y_p solves the inhomogeneous DE, we have $(D^2-4)y_p = 3e^x + 5x^2$. The right-hand side $3e^x + 5x^2$ is a solution of $p(D)y = 0$ if and only if $1, 0, 0, 0$ are roots of the characteristic polynomial $p(D)$. In particular, $(D-1)D^3(3e^x + 5x^2) = 0$. Combined, we find that $(D-1)D^3(D^2-4)y_p = 0$.

(c) $(N-4)^2(N-1)a_n = 0$

Explanation. $a_n = (n+3)4^n + 5 \cdot 1^n$ is a solution of $p(N)a_n = 0$ if and only if 4 (repeated two times) and 1 are a root of the characteristic polynomial $p(N)$. Hence, the simplest recurrence is obtained from $p(N) = (N-4)^2(N-1)$.

[Since, $(N-4)^2(N-1) = N^3 - 9N^2 + 24N - 16$, the recurrence in explicit form is $a_{n+3} = 9a_{n+2} - 24a_{n+1} + 16a_n$.]

(d) $M^n = \begin{bmatrix} 2 \cdot 2^n - 1 & -2 \cdot 2^n + 2 \\ 2^n - 1 & -2^n + 2 \end{bmatrix}$

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Problem 4. (3 points) Consider the following system of initial value problems:

$$\begin{aligned} y_1'' &= 4y_1 + 3y_2 \\ y_2'' &= 5y_1' - 2y_2' \end{aligned} \qquad y_1(0) = 1,\ y_1'(0) = 0,\ y_2(0) = 6,\ y_2'(0) = 7$$

Write it as a first-order initial value problem in the form $\boldsymbol{y}' = M\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{c}$.

Solution. Introduce $y_3 = y_1'$ and $y_4 = y_2'$. Then, the given system translates into

$$\boldsymbol{y}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 4 & 3 & 0 & 0 \\ 0 & 0 & 5 & -2 \end{bmatrix} \boldsymbol{y}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 6 \\ 0 \\ 7 \end{bmatrix}.$$

Problem 5. (1+4+1 points) Consider the sequence a_n defined by $a_{n+2} = a_{n+1} + 2a_n$ and $a_0 = 0$, $a_1 = 6$.

(a) The next two terms are $a_2 =$ [ ] and $a_3 =$ [ ].

(b) A Binet-like formula for a_n is $a_n =$ [ ], and $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} =$ [ ].

Solution.

(a) $a_2 = 6$, $a_3 = 18$

(b) The recursion can be written as $p(N)a_n = 0$ where $p(N) = N^2 - N - 2$ has roots $2, -1$.

Hence, $a_n = C_1\, 2^n + C_2\, (-1)^n$ and we only need to figure out the two unknowns C_1, C_2. We can do that using the two initial conditions: $a_0 = C_1 + C_2 = 0$, $a_1 = 2C_1 - C_2 = 6$.

Solving, we find $C_1 = 2$ and $C_2 = -2$ so that, in conclusion, $a_n = 2 \cdot 2^n - 2(-1)^n$.

(c) It follows from the Binet-like formula that $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 2$.

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(extra scratch paper)

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