A crash course in linear algebra

Example 1. A typical 2×3 matrix is $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

It is composed of column vectors like $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and row vectors like $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$.

Matrices (and vectors) of the same dimensions can be added and multiplied by a scalar:

For instance, $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 5 \\ 6 & 8 & 5 \end{bmatrix}$ or $3\cdot \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \end{bmatrix}$.

Remark. More generally, a **vector space** is an abstraction of a collection of objects that can be added and scaled: numbers, lists of numbers (like the above row and column vectors), arrays of numbers (like the above matrices), arrows, functions, polynomials, differential operators, solutions to homogeneous linear differential equations, $\ldots$

Example 2. The **transpose** A^T of A is obtained by interchanging roles of rows and columns.

For instance. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

Example 3. Matrices of appropriate dimensions can also be **multiplied**.

This is based on the multiplication $\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = ax + by + cz$ of row and column vectors.

For instance. $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 7 & -5 \end{bmatrix}$

In general, we can multiply a $m\times n$ matrix A with a $n\times r$ matrix B to get a $m\times r$ matrix AB.

Its entry in row i and column j is defined to be $(AB)_{ij} = (\text{row } i \text{ of } A) \begin{bmatrix} \text{column} \\ j \\ \text{of } B \end{bmatrix}$.

Comment. One way to think about the multiplication $A\boldsymbol{x}$ is that the resulting vector is a linear combination of the columns of A with coefficients from $\boldsymbol{x}$. Similarly, we can think of $\boldsymbol{x}^T A$ as a combination of the rows of A.

Some nice properties of matrix multiplication are:

- There is an $n\times n$ identity matrix I (all entries are zero except the diagonal ones which are 1). It satisfies $AI = A$ and $IA = A$.
- The associative law $A(BC) = (AB)C$ holds. Hence, we can write ABC without ambiguity.
- The distributive laws including $A(B+C) = AB + AC$ hold.

Example 4. $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \neq \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, so we have no commutative law.

Example 5. $\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

On the RHS we have the **identity matrix**, usually denoted I or I_2 (since it's the 2×2 identity matrix here).

Hence, the two matrices on the left are inverses of each other: $\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$, $\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$.

Armin Straub
straub@southalabama.edu

The **inverse** A^{-1} of a matrix A is characterized by $A^{-1}A = I$ and $AA^{-1} = I$.

Example 6. The following formula immediately gives us the inverse of a 2×2 matrix (if it exists). It is worth remembering!

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad \text{provided that } ad-bc \neq 0$$

Let's check that! $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} ad-bc & 0 \\ 0 & -cb+ad \end{bmatrix} = I_2$

In particular, a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible $\iff ad-bc \neq 0$.

Recall that this is the **determinant**: $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad-bc$.

$$\det(A) = 0 \quad \iff \quad A \text{ is not invertible}$$

Example 7. The system $\begin{aligned} 7x_1 - 2x_2 &= 3 \\ 2x_1 + x_2 &= 4 \end{aligned}$ is equivalent to $\begin{bmatrix} 7 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. Solve it.

Solution. Multiplying (from the left!) by $\begin{bmatrix} 7 & -2 \\ 2 & 1 \end{bmatrix}^{-1} = \frac{1}{11}\begin{bmatrix} 1 & 2 \\ -2 & 7 \end{bmatrix}$ produces $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 1 & 2 \\ -2 & 7 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, which gives the solution of the original equations.

Example 8. (homework) Solve the system $\begin{aligned} x_1 + 2x_2 &= 1 \\ 3x_1 + 4x_2 &= -1 \end{aligned}$ (using a matrix inverse).

Solution. The equations are equivalent to $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Multiplying by $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = -\frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$ produces $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 6 \\ -4 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$.

Example 9. (homework) Solve the system $\begin{aligned} x_1 + 2x_2 &= 1 \\ 3x_1 + 4x_2 &= 2 \end{aligned}$ (using a matrix inverse).

Solution. The equations are equivalent to $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Multiplying by $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = -\frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$ produces $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$.

Comment. In hindsight, can you see this solution by staring at the equations?

Comment. Note how we can reuse the matrix inverse from the previous example.

The **determinant** of A, written as $\det(A)$ or $|A|$, is a number with the property that:

$$\begin{aligned} \det(A) \neq 0 &\iff A \text{ is invertible} \\ &\iff A\boldsymbol{x} = \boldsymbol{b} \text{ has a (unique) solution } \boldsymbol{x} \text{ for all } \boldsymbol{b} \\ &\iff A\boldsymbol{x} = 0 \text{ is only solved by } \boldsymbol{x} = 0 \end{aligned}$$

Example 10. $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad-bc$, which appeared in the formula for the inverse.

Armin Straub
straub@southalabama.edu

Example 11. (review) $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 14 \end{bmatrix}$ whereas $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$.

Review: Examples of differential equations we can solve

Let's start with one of the simplest (and most fundamental) differential equations (DE). It is **first-order** (only a first derivative) and **linear** (with constant coefficients).

Example 12. Solve $y' = 3y$.

Solution. $y(x) = Ce^{3x}$

Check. Indeed, if $y(x) = Ce^{3x}$, then $y'(x) = 3Ce^{3x} = 3y(x)$.

Comment. Recall we can always easily check whether a function solves a differential equation. This means that (although you might be unfamiliar with the techniques for solving) you can use computer algebra systems like Sage to solve differential equations without trust issues.

To describe a unique solution, additional constraints need to be imposed.

Example 13. Solve the **initial value problem** (IVP) $y' = 3y$, $y(0) = 5$.

Solution. This has the unique solution $y(x) = 5e^{3x}$.

The following is a **non-linear** differential equation. In general, such equations are much more complicated than linear ones. We can solve this particular one because it is **separable**.

Example 14. Solve $y' = xy^2$.

Solution. This DE is separable: $\frac{1}{y^2}\mathrm{d}y = x\,\mathrm{d}x$. Integrating, we find $-\frac{1}{y} = \frac{1}{2}x^2 + C$.

Hence, $y = -\frac{1}{\frac{1}{2}x^2 + C} = \frac{2}{D - x^2}$.

[Here, $D = -2C$ but that relationship doesn't matter; it only matters that the solution has a free parameter.]

Note. Note that we did not find the solution $y = 0$ (lost when dividing by y^2). It is called a **singular solution** because it is not part of the **general solution** (the one-parameter family found above). [Although, we can obtain it from the general solution by letting $D \to \infty$.]

Check. Compute y' and verify that the DE is indeed satisfied.

Review: Linear first-order DEs

The most general first-order linear DE is $P(x)y' + Q(x)y + R(x) = 0$.

By dividing by $P(x)$ and rearranging, we can always write it in the form $y' = a(x)y + f(x)$.

We will recall next time that we can always solve this DE.

The corresponding **homogeneous** linear DE is $y' = a(x)y$.

Important comment. Write $D = \frac{\mathrm{d}}{\mathrm{d}x}$. Then we can write $y' - a(x)y = f(x)$ as $Ly = f(x)$ where $L = D - a(x)$. The corresponding homogeneous DE is simply $Ly = 0$.

Armin Straub
straub@southalabama.edu

Solving linear first-order DEs using variation of constants

The following DE is linear and first-order (but not with constant coefficients).

Example 15. (homework) Solve $y' = x^2 y$.

Solution. This DE is separable as well: $\frac{1}{y}\mathrm{d}y = x^2\,\mathrm{d}x$ (note that we just lost the solution $y = 0$).

Integrating gives $\ln|y| = \frac{1}{3}x^3 + A$, so that $|y| = e^{\frac{1}{3}x^2 + A}$. Since the RHS is never zero, we must have either $y = e^{\frac{1}{3}x^2 + A}$ or $y = -e^{\frac{1}{3}x^2 + A}$.

Hence $y = \pm e^A e^{\frac{1}{3}x^3} = Ce^{\frac{1}{3}x^3}$ (with $C = \pm e^A$). Note that $C = 0$ corresponds to the singular solution $y = 0$.

In summary, the general solution is $y = Ce^{\frac{1}{3}x^3}$ (with C any real number).

Check. Compute y' and verify that the DE is indeed satisfied.

As in the previous example, we can immediately solve any **homogeneous** linear first-order DE:

Example 16. Solve $y' = a(x)y$.

Solution. Proceeding as in the previous example, we find $y(x) = Ce^{\int a(x)\mathrm{d}x}$.

Check. Compute y' and verify that the DE is indeed satisfied.

Review. Every **homogeneous linear first-order DE** can be written as $y' = a(x)y$.

Its general solution is $y(x) = Ce^{\int a(x)\mathrm{d}x}$.

Comment. Note that the constant of integration can be absorbed into the factor C (in other words, there is only one degree of freedom in the above formula for the general solution).

Example 17. Solve $y' = x^2 y$.

Solution. This is a homogeneous linear first-order DE $y' = a(x)y$ with $a(x) = x^2$. Accordingly, the general solution is $y(x) = Ce^{\int a(x)\mathrm{d}x} = Ce^{\int x^2\mathrm{d}x} = Ce^{\frac{1}{3}x^3}$.

Comment. As noted above, the constant of integration gets absorbed into the factor C.

Comment. Alternatively, we can solve this DE via separation of variables (see earlier example).

Recall that, to find the general solution of the **inhomogeneous DE**

$$y' = a(x)y + f(x),$$

we only need to find a particular solution y_p.

Then the general solution is $y_p + Cy_h$, where y_h is any solution of the homogeneous DE $y' = a(x)y$.

Comment. In applications, $f(x)$ often represents an external force. As such, the inhomogeneous DE is sometimes called "driven" while the homogeneous DE would be called "undriven".

Theorem 18. (variation of constants) $y' = a(x)y + f(x)$ has the particular solution

$$y_p(x) = c(x)y_h(x) \quad \text{with } c(x) = \int \frac{f(x)}{y_h(x)}\mathrm{d}x,$$

where $y_h(x) = e^{\int a(x)\mathrm{d}x}$ is any solution to the homogeneous equation $y' = a(x)y$.

Proof. Let us plug $y_p(x) = y_h(x)\int \frac{f(x)}{y_h(x)}\mathrm{d}x$ into the DE to verify that it is a solution:

$$y_p'(x) = y_h'(x)\int \frac{f(x)}{y_h(x)}\mathrm{d}x + y_h(x)\underbrace{\frac{\mathrm{d}}{\mathrm{d}x}\int \frac{f(x)}{y_h(x)}\mathrm{d}x}_{\frac{f(x)}{y_h(x)}} = a(x)y_h(x)\int \frac{f(x)}{y_h(x)}\mathrm{d}x + f(x) = a(x)y_p(x) + f(x) \qquad \square$$

Comment. Note that the formula for $y_p(x)$ gives the general solution if we let $\int \frac{f(x)}{y_h(x)}\mathrm{d}x$ be the general antiderivative. (Think about the effect of the constant of integration!)

Recall. The formula for $y_p(x)$ can be found using **variation of constants** (sometimes called variation of parameters): that is, we look for solutions of the form $y(x) = c(x)y_h(x)$.

If we plug $y(x) = c(x)y_h(x)$ into the DE $y' = ay + f$, we find $c'y_h + cy_h' = acy_h + f$. Since $y_h' = ay_h$, this simplifies to $c'y_h = f$ or, equivalently, $c' = \frac{f}{y_h}$.

Hence, $c(x) = \int \frac{f(x)}{y_h(x)}\mathrm{d}x$, which is the formula in the theorem.

Armin Straub
straub@southalabama.edu

Example 19. Solve $x^2y' = 1 - xy + 2x$, $y(1) = 3$.

Solution. Write as $\frac{\mathrm{d}y}{\mathrm{d}x} = a(x)y + f(x)$ with $a(x) = -\frac{1}{x}$ and $f(x) = \frac{1}{x^2} + \frac{2}{x}$.

$y_h(x) = e^{\int a(x)\mathrm{d}x} = e^{-\ln x} = \frac{1}{x}$. (Why can we write $\ln x$ instead of $\ln|x|$? See comment below.) Hence:

$$y_p(x) = y_h(x)\int \frac{f(x)}{y_h(x)}\mathrm{d}x = \frac{1}{x}\int \left(\frac{1}{x} + 2\right)\mathrm{d}x = \frac{\ln x + 2x + C}{x}$$

Using $y(1) = 3$, we find $C = 1$. In summary, the solution is $y = \frac{\ln(x) + 2x + 1}{x}$.

Comment. Note that $x = 1 > 0$ in the initial condition. Because of that we know that (at least locally) our solution will have $x > 0$. Accordingly, we can use $\ln x$ instead of $\ln|x|$. (If the initial condition had been $y(-1) = 3$, then we would have $x < 0$, in which case we can use $\ln(-x)$ instead of $\ln|x|$.)

Comment. Observe how the general solution (with parameter C) is indeed obtained from any particular solution (say, $\frac{\ln x + 2x}{x}$) plus the general solution to the homogeneous equation, which is $\frac{C}{x}$.

Armin Straub
straub@southalabama.edu

Review: Linear DEs

A linear DE of order n is of the form $y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \ldots + p_1(x) y' + p_0(x) y = f(x)$.

- In terms of $D = \frac{\mathrm{d}}{\mathrm{d}x}$, the DE becomes: $L y = f(x)$ with $L = D^n + p_{n-1}(x) D^{n-1} + \ldots + p_1(x) D + p_0(x)$.
  Comment. L is called a (linear) differential operator.
- The inclusion of the $f(x)$ term makes $L y = f(x)$ an **inhomogeneous** linear DE.
- $L y = 0$ is the corresponding **homogeneous** DE.
  - If y_1 and y_2 are solutions to the homogeneous DE, then so is any linear combination $C_1 y_1 + C_2 y_2$.
  - **(general solution of the homogeneous DE)** There are n solutions $y_1, y_2, \ldots, y_n$, such that every solution is of the form $C_1 y_1 + \ldots + C_n y_n$. [These n solutions necessarily are **independent**.]
- To find the general solution of the inhomogeneous DE, we only need to find a single solution y_p (called a **particular solution**). Then the general solution is $y_p + y_h$, where y_h is the general solution of the homogeneous DE.

Example 20. Consider the following DEs. If linear, write them in operator form as $L y = f(x)$.

(a) $y'' = x y$

(b) $x^2 y'' + x y' = (x^2 + 4) y + x(x^2 + 3)$

(c) $y'' = y' + 2y + 2(1 - x - x^2)$

(d) $y'' = y' + 2y + 2(1 - x - y^2)$

Solution.

(a) This is a homogeneous linear DE: $\underbrace{(D^2 - x)}_{L} y = \underbrace{0}_{f(x)}$

Note. This is known as the Airy equation, which we will meet again later. The general solution is of the form $C_1 y_1(x) + C_2 y_2(x)$ for two special solutions y_1, y_2. [In the literature, one usually chooses functions called $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ as y_1 and y_2. See: https://en.wikipedia.org/wiki/Airy_function]

(b) This is an inhomogeneous linear DE: $\underbrace{(x^2 D^2 + x D - (x^2 + 4))}_{L} y = \underbrace{x(x^2 + 3)}_{f(x)}$

Note. The corresponding homogeneous DE is an instance of the "modified Bessel equation" $x^2 y'' + x y' - (x^2 + \alpha^2) y = 0$, namely the case $\alpha = 2$. Because they are important for applications (but cannot be written in terms of familiar functions), people have introduced names for two special solutions of this differential equation: $I_\alpha(x)$ and $K_\alpha(x)$ (called modified Bessel functions of the first and second kind).

It follows that the general solution of the modified Bessel equation is $C_1 I_\alpha(x) + C_2 K_\alpha(x)$.

In our case. The general solution of the homogeneous DE (which is the modified Bessel equation with $\alpha = 2$) is $C_1 I_2(x) + C_2 K_2(x)$. On the other hand, we can (do it!) easily check (this is coming from nowhere at this point!) that $y_p = -x$ is a particular solution to the original inhomogeneous DE.

It follows that the general solution to the original DE is $C_1 I_2(x) + C_2 K_2(x) - x$.

(c) This is an inhomogeneous linear DE: $\underbrace{(D^2 - D - 2)}_{L} y = \underbrace{2(1 - x - x^2)}_{f(x)}$

Note. We will recall in Example 21 that the corresponding homogeneous DE $(D^2 - D - 2) y = 0$ has general solution $C_1 e^{2x} + C_2 e^{-x}$. On the other hand, we can check that $y_p = x^2$ is a particular solution of the original inhomogeneous DE. (Do you recall from DE1 how to find this particular solution?)

It follows that the general solution to the original DE is $x^2 + C_1 e^{2x} + C_2 e^{-x}$.

(d) This is not a linear DE because of the term y^2. It cannot be written in the form $L y = f(x)$.

Homogeneous linear DEs with constant coefficients

Example 21. Find the general solution to $y'' - y' - 2y = 0$.

Solution. We recall from *Differential Equations I* that e^{rx} solves this DE for the right choice of r.
Plugging e^{rx} into the DE, we get $r^2 e^{rx} - r e^{rx} - 2e^{rx} = 0$.
Equivalently, $r^2 - r - 2 = 0$. This is called the **characteristic equation**. Its solutions are $r = 2, -1$.
This means we found the two solutions $y_1 = e^{2x}$, $y_2 = e^{-x}$.
Since this a homogeneous linear DE, the general solution is $y = C_1 e^{2x} + C_2 e^{-x}$.

Solution. (operators) $y'' - y' - 2y = 0$ is equivalent to $(D^2 - D - 2)y = 0$.
Note that $D^2 - D - 2 = (D-2)(D+1)$ is the **characteristic polynomial**.
It follows that we get solutions to $(D-2)(D+1)y = 0$ from $(D-2)y = 0$ and $(D+1)y = 0$.
$(D-2)y = 0$ is solved by $y_1 = e^{2x}$, and $(D+1)y = 0$ is solved by $y_2 = e^{-x}$; as in the previous solution.

Example 22. Solve $y'' - y' - 2y = 0$ with initial conditions $y(0) = 4$, $y'(0) = 5$.

Solution. From the previous example, we know that $y(x) = C_1 e^{2x} + C_2 e^{-x}$.
To match the initial conditions, we need to solve $C_1 + C_2 = 4$, $2C_1 - C_2 = 5$. We find $C_1 = 3$, $C_2 = 1$.
Hence the solution is $y(x) = 3e^{2x} + e^{-x}$.

Set $D = \frac{\mathrm{d}}{\mathrm{d}x}$. Every **homogeneous linear DE with constant coefficients** can be written as $p(D)y = 0$, where $p(D)$ is a polynomial in D, called the **characteristic polynomial**.

For instance. $y'' - y' - 2y = 0$ is equivalent to $Ly = 0$ with $L = D^2 - D - 2$.

Example 23. Find the general solution of $y''' + 7y'' + 14y' + 8y = 0$.

Solution. This DE is of the form $p(D)\,y = 0$ with characteristic polynomial $p(D) = D^3 + 7D^2 + 14D + 8$.
The characteristic polynomial factors as $p(D) = (D+1)(D+2)(D+4)$. (Don't worry! You won't be asked to factor cubic polynomials by hand.)
Hence, by the same argument as in Example 21, we find the solutions $y_1 = e^{-x}$, $y_2 = e^{-2x}$, $y_3 = e^{-4x}$. That's enough (independent!) solutions for a third-order DE.
The general solution therefore is $y(x) = C_1\, e^{-x} + C_2\, e^{-2x} + C_3\, e^{-4x}$.

This approach applies to any homogeneous linear DE with constant coefficients!

One issue is that roots might be repeated. In that case, we are currently missing solutions. The following result provides the missing solutions.

Theorem 24. Consider the homogeneous linear DE with constant coefficients $p(D)y = 0$.

- If r is a root of the characteristic polynomial and if k is its multiplicity, then k (independent) solutions of the DE are given by $x^j e^{rx}$ for $j = 0, 1, \ldots, k-1$.
- Combining these solutions for all roots, gives the general solution.

This is because the order of the DE equals the degree of $p(D)$, and a polynomial of degree n has (counting with multiplicity) exactly n (possibly **complex**) roots.

In the complex case. If $r = a \pm bi$ are roots of the characteristic polynomial and if k is its multiplicity, then $2k$ (independent) **real solutions** of the DE are given by $x^j e^{ax}\cos(bx)$ and $x^j e^{ax}\sin(bx)$ for $j = 0, 1, \ldots, k-1$.

Armin Straub
straub@southalabama.edu

Proof. Let r be a root of the characteristic polynomial of multiplicity k. Then $p(D)=q(D)\,(D-r)^k$.
We need to find k solutions to the simpler DE $(D-r)^k y=0$.
It is natural to look for solutions of the form $y=c(x)e^{rx}$.
[We know that $c(x)=1$ provides a solution. Note that this is the same idea as for variation of constants.]
Note that $(D-r)[c(x)e^{rx}]=(c'(x)e^{rx}+c(x)re^{rx})-rc(x)e^{rx}=c'(x)e^{rx}$.
Repeating, we get $(D-r)^2[c(x)e^{rx}]=(D-r)[c'(x)e^{rx}]=c''(x)e^{rx}$ and, eventually, $(D-r)^k[c(x)e^{rx}]=c^{(k)}(x)e^{rx}$.
In particular, $(D-r)^k y=0$ is solved by $y=c(x)e^{rx}$ if and only if $c^{(k)}(x)=0$.
The DE $c^{(k)}(x)=0$ is clearly solved by x^j for $j=0,1,\ldots,k-1$, and it follows that $x^j e^{rx}$ solves the original DE. □

Example 25. Find the general solution of $y'''=0$.

Solution. We know from Calculus that the general solution is $y(x)=C_1+C_2x+C_3x^2$.

Solution. The characteristic polynomial $p(D)=D^3$ has roots $0,0,0$. By Theorem 24, we have the solutions $y(x)=x^j e^{0x}=x^j$ for $j=0,1,2$, so that the general solution is $y(x)=C_1+C_2x+C_3x^2$.

Example 26. Find the general solution of $y'''-y''-5y'-3y=0$.

Solution. The characteristic polynomial $p(D)=D^3-D^2-5D-3=(D-3)(D+1)^2$ has roots $3,-1,-1$.
By Theorem 24, the general solution is $y(x)=C_1e^{3x}+(C_2+C_3x)e^{-x}$.

Example 27. Find the general solution of $y''+y=0$.

Solution. The characteristic polynomials is $p(D)=D^2+1=0$ which has no solutions over the reals.
Over the **complex numbers**, by definition, the roots are i and $-i$.
So the general solution is $y(x)=C_1e^{ix}+C_2e^{-ix}$.

Solution. On the other hand, we easily check that $y_1=\cos(x)$ and $y_2=\sin(x)$ are two solutions.
Hence, the general solution can also be written as $y(x)=D_1\cos(x)+D_2\sin(x)$.

Important comment. That we have these two different representations is a consequence of **Euler's identity**

$$e^{ix}=\cos(x)+i\sin(x).$$

Note that $e^{-ix}=\cos(x)-i\sin(x)$.
On the other hand, $\cos(x)=\frac{1}{2}(e^{ix}+e^{-ix})$ and $\sin(x)=\frac{1}{2i}(e^{ix}-e^{-ix})$.
[Recall that the first formula is an instance of $\mathrm{Re}(z)=\frac{1}{2}(z+\bar{z})$ and the second of $\mathrm{Im}(z)=\frac{1}{2i}(z-\bar{z})$.]

Example 28. Find the general solution of $y''-4y'+13y=0$.

Solution. The characteristic polynomial $p(D)=D^2-4D+13$ has roots $2+3i,2-3i$.
Hence, the general solution is $y(x)=C_1e^{2x}\cos(3x)+C_2e^{2x}\sin(3x)$.
Note. $e^{(2+3i)x}=e^{2x}e^{3ix}=e^{2x}(\cos(3x)+i\sin(3x))$

Example 29. (review) Find the general solution of $y''' - 3y' + 2y = 0$.

Solution. The characteristic polynomial $p(D) = D^3 - 3D + 2 = (D-1)^2(D+2)$ has roots $1, 1, -2$.

By Theorem 24, the general solution is $y(x) = (C_1 + C_2x)e^x + C_2e^{-2x}$.

Inhomogeneous linear DEs with constant coefficients

Example 30. ("warmup") Find the general solution of $y'' + 4y = 12x$.

Solution. Here, $p(D) = D^2 + 4$, which has roots $\pm 2i$.

Hence, the general solution is $y(x) = y_p(x) + C_1\cos(2x) + C_2\sin(2x)$. It remains to find a particular solution y_p.

Noting that $D^2 \cdot (12x) = 0$, we apply D^2 to both sides of the DE.

We get $D^2(D^2 + 4) \cdot y = 0$, which is a homogeneous linear DE! Its general solution is $C_1 + C_2x + C_3\cos(2x) + C_4\sin(2x)$. In particular, y_p is of this form for some choice of $C_1, \ldots, C_4$.

It simplifies our life to note that there has to be a particular solution of the simpler form $y_p = C_1 + C_2x$.

[Why?! Because we know that $C_3\cos(2x) + C_4\sin(2x)$ can be added to any particular solution.]

It only remains to find appropriate values C_1, C_2 such that $y_p'' + 4y_p = 12x$. Since $y_p'' + 4y_p = 4C_1 + 4C_2x$, comparing coefficients yields $4C_1 = 0$ and $4C_2 = 12$, so that $C_1 = 0$ and $C_2 = 3$. In other words, $y_p = 3x$.

Therefore, the general solution to the original DE is $y(x) = 3x + C_1\cos(2x) + C_2\sin(2x)$.

Example 31. ("warmup") Find the general solution of $y'' + 4y' + 4y = e^{3x}$.

Solution. This is $p(D)y = e^{3x}$ with $p(D) = D^2 + 4D + 4 = (D+2)^2$.

Hence, the general solution is $y(x) = y_p(x) + (C_1 + C_2x)e^{-2x}$. It remains to find a particular solution y_p.

Note that $(D-3)e^{3x} = 0$. Hence, we apply $(D-3)$ to the DE to get $(D-3)(D+2)^2y = 0$.

This homogeneous linear DE has general solution $(C_1 + C_2x)e^{-2x} + C_3e^{3x}$. We conclude that the original DE must have a particular solution of the form $y_p = C_3e^{3x}$.

To determine the value of C_3, we plug into the original DE: $y_p'' + 4y_p' + 4y_p = (9 + 4\cdot 3 + 4)C_3e^{3x} \overset{!}{=} e^{3x}$. Hence, $C_3 = 1/25$. In conclusion, the general solution is $y(x) = (C_1 + C_2x)e^{-2x} + \frac{1}{25}e^{3x}$.

Comment. See Example 33 for the same solution in more compact form.

We found a recipe for solving nonhomogeneous linear DEs with constant coefficients.

Our approach works for $p(D)y = f(x)$ whenever the right-hand side $f(x)$ is the solution of some homogeneous linear DE with constant coefficients: $q(D)f(x) = 0$

Theorem 32. (method of undetermined coefficients) To find a particular solution y_p to an inhomogeneous linear DE with constant coefficients $p(D)y = f(x)$:

- Find $q(D)$ so that $q(D)f(x) = 0$. [This does not work for all $f(x)$.]
- Let $r_1, \ldots, r_n$ be the ("old") roots of the polynomial $p(D)$.
  Let $s_1, \ldots, s_m$ be the ("new") roots of the polynomial $q(D)$.
- It follows that y_p solves the **homogeneous** DE $q(D)\,p(D)y = 0$.
  The characteristic polynomial of this DE has roots $r_1, \ldots, r_n, s_1, \ldots, s_m$.
  Let $v_1, \ldots, v_m$ be the "new" solutions (i.e. not solutions of the "old" $p(D)y = 0$).
  By plugging into $p(D)y_p = f(x)$, we find (unique) C_i so that $y_p = C_1v_1 + \ldots + C_mv_m$.

Because of the final step, this approach is often called **method of undetermined coefficients**.

For which $f(x)$ does this work? By Theorem 24, we know exactly which $f(x)$ are solutions to homogeneous linear DEs with constant coefficients: these are linear combinations of exponentials $x^j\,e^{rx}$ (which includes $x^j\,e^{ax}\cos(bx)$ and $x^j\,e^{ax}\sin(bx)$).

Armin Straub
straub@southalabama.edu

Example 33. (again) Determine the general solution of $y'' + 4y' + 4y = e^{3x}$.

Solution. The "old" roots are $-2, -2$. The "new" roots are 3. Hence, there has to be a particular solution of the form $y_p = Ce^{3x}$. To find the value of C, we plug into the DE.

$y_p'' + 4y_p' + 4y_p = (9 + 4 \cdot 3 + 4)Ce^{3x} \overset{!}{=} e^{3x}$. Hence, $C = 1/25$.

Therefore, the general solution is $y(x) = (C_1 + C_2 x)e^{-2x} + \frac{1}{25}e^{3x}$.

Example 34. Determine the general solution of $y'' + 4y' + 4y = 7e^{-2x}$.

Solution. The "old" roots are $-2, -2$. The "new" roots are -2. Hence, there has to be a particular solution of the form $y_p = Cx^2 e^{-2x}$. To find the value of C, we plug into the DE.

$y_p' = C(-2x^2 + 2x)e^{-2x}$

$y_p'' = C(4x^2 - 8x + 2)e^{-2x}$

$y_p'' + 4y_p' + 4y_p = 2Ce^{-2x} \overset{!}{=} 7e^{-2x}$

It follows that $C = 7/2$, so that $y_p = \frac{7}{2}x^2 e^{-2x}$. The general solution is $y(x) = \left(C_1 + C_2 x + \frac{7}{2}x^2\right)e^{-2x}$.

Example 35. Determine a particular solution of $y'' + 4y' + 4y = 2e^{3x} - 5e^{-2x}$.

Solution. Write the DE as $Ly = 2e^{3x} - 5e^{-2x}$ where $L = D^2 + 4D + 4$. Instead of starting all over, recall that in Example 33 we found that $y_1 = \frac{1}{25}e^{3x}$ satisfies $Ly_1 = e^{3x}$. Also, in Example 34 we found that $y_2 = \frac{7}{2}x^2 e^{-2x}$ satisfies $Ly_2 = 7e^{-2x}$.

By linearity, it follows that $L(Ay_1 + By_2) = ALy_1 + BLy_2 = Ae^{3x} + 7Be^{-2x}$.

To get a particular solution y_p of our DE, we need $A = 2$ and $7B = -5$.

Hence, $y_p = 2y_1 - \frac{5}{7}y_2 = \frac{2}{25}e^{3x} - \frac{5}{2}x^2 e^{-2x}$.

Example 36. (homework) Determine the general solution of $y'' - 2y' + y = 5\sin(3x)$.

Solution. Since $D^2 - 2D + 1 = (D-1)^2$, the "old" roots are $1, 1$. The "new" roots are $\pm 3i$. Hence, there has to be a particular solution of the form $y_p = A\cos(3x) + B\sin(3x)$.

To find the values of A and B, we plug into the DE.

$y_p' = -3A\sin(3x) + 3B\cos(3x)$

$y_p'' = -9A\cos(3x) - 9B\sin(3x)$

$y_p'' - 2y_p' + y_p = (-8A - 6B)\cos(3x) + (6A - 8B)\sin(3x) \overset{!}{=} 5\sin(3x)$

Equating the coefficients of $\cos(x)$, $\sin(x)$, we obtain the two equations $-8A - 6B = 0$ and $6A - 8B = 5$.

Solving these, we find $A = \frac{3}{10}$, $B = -\frac{2}{5}$. Accordingly, a particular solution is $y_p = \frac{3}{10}\cos(3x) - \frac{2}{5}\sin(3x)$.

The general solution is $y(x) = \frac{3}{10}\cos(3x) - \frac{2}{5}\sin(3x) + (C_1 + C_2 x)e^x$.

Example 37. (homework) What is the shape of a particular solution of $y'' + 4y' + 4y = x\cos(x)$?

Solution. The "old" roots are $-2, -2$. The "new" roots are $\pm i, \pm i$. Hence, there has to be a particular solution of the form $y_p = (C_1 + C_2 x)\cos(x) + (C_3 + C_4 x)\sin(x)$.

Continuing to find a particular solution. To find the value of the C_j's, we plug into the DE.

$y_p' = (C_2 + C_3 + C_4 x)\cos(x) + (C_4 - C_1 - C_2 x)\sin(x)$

$y_p'' = (2C_4 - C_1 - C_2 x)\cos(x) + (-2C_2 - C_3 - C_4 x)\sin(x)$

$$y_p'' + 4y_p' + 4y_p = (3C_1 + 4C_2 + 4C_3 + 2C_4 + (3C_2 + 4C_4)x)\cos(x)$$
$$+ (-4C_1 - 2C_2 + 3C_3 + 4C_4 + (-4C_2 + 3C_4)x)\sin(x) \overset{!}{=} x\cos(x).$$

Equating the coefficients of $\cos(x)$, $x\cos(x)$, $\sin(x)$, $x\sin(x)$, we get the equations $3C_1 + 4C_2 + 4C_3 + 2C_4 = 0$, $3C_2 + 4C_4 = 1$, $-4C_1 - 2C_2 + 3C_3 + 4C_4 = 0$, $-4C_2 + 3C_4 = 0$.

Solving (this is tedious!), we find $C_1 = -\frac{4}{125}$, $C_2 = \frac{3}{25}$, $C_3 = -\frac{22}{125}$, $C_4 = \frac{4}{25}$.

Hence, $y_p = \left(-\frac{4}{125} + \frac{3}{25}x\right)\cos(x) + \left(-\frac{22}{125} + \frac{4}{25}x\right)\sin(x)$.

Example 38. (review) What is the shape of a particular solution of $y'' + 4y' + 4y = 4e^{3x}\sin(2x) - x\sin(x)$?

Solution. The "old" roots are $-2, -2$. The "new" roots are $3 \pm 2i, \pm i, \pm i$.

Hence, there has to be a particular solution of the form

$y_p = C_1 e^{3x}\cos(2x) + C_2 e^{3x}\sin(2x) + (C_3 + C_4 x)\cos(x) + (C_5 + C_6 x)\sin(x)$.

Continuing to find a particular solution. To find the values of $C_1, \ldots, C_6$, we plug into the DE. But this final step is so boring that we don't go through it here. Computers (currently?) cannot afford to be as selective; mine obediently calculated: $y_p = -\frac{4}{841}e^{3x}(20\cos(2x) - 21\sin(2x)) + \frac{1}{125}((-22 + 20x)\cos(x) + (4 - 15x)\sin(x))$

Sage

In practice, we are happy to let a machine do tedious computations. Let us see how to use the open-source computer algebra system **Sage** to do basic computations for us.

Sage is freely available at `sagemath.org`. Instead of installing it locally (it's huge!) we can conveniently use it in the cloud at `cocalc.com` from any browser.

[For basic computations, you can also simply use the textbox on our course website.]

Sage is built as a **Python** library, so any Python code is valid. For starters, we will use it as a fancy calculator.

Example 39. To solve the differential equation $y'' + 4y' + 4y = 7e^{-2x}$, as we did in Example 34, we can use the following:

```
>>> x = var('x')
>>> y = function('y')(x)
>>> desolve(diff(y,x,2) + 4*diff(y,x) + 4*y == 7*exp(-2*x), y)
```

$$\frac{7}{2}\, x^2\, e^{(-2\,x)} + (K_2\, x + K_1)\, e^{(-2\,x)}$$

This confirms, as we had found, that the general solution is $y(x) = \left(C_1 + C_2 x + \frac{7}{2}x^2\right)e^{-2x}$.

Example 40. Similarly, Sage can solve initial value problems such as $y'' - y' - 2y = 0$ with initial conditions $y(0) = 4$, $y'(0) = 5$.

```
>>> x = var('x')
>>> y = function('y')(x)
>>> desolve(diff(y,x,2) - diff(y,x) - 2*y == 0, y, ics=[0,4,5])
```

$$3\, e^{(2\,x)} + e^{(-x)}$$

This matches the (unique) solution $y(x) = 3e^{2x} + e^{-x}$ that we derived in Example 22.

Armin Straub
straub@southalabama.edu

Example 41. We have been factoring differential operators like $D^2+4D+4=(D+2)^2$.

Things become much more complicated when the coefficients are not constant!

For instance, the linear DE $y''+4y'+4xy=0$ can be written as $Ly=0$ with $L=D^2+4D+4x$. However, in general, such operators cannot be factored (unless we allow as coefficients functions in x that we are not familiar with). [On the other hand, any ordinary polynomial can be factored over the complex numbers.]

One indication that things become much more complicated is that x and D do not commute: $xD \neq Dx$!!

Indeed, $(xD)f(x)=xf'(x)$ while $(Dx)f(x)=\frac{\mathrm{d}}{\mathrm{d}x}[xf(x)]=f(x)+xf'(x)=(1+xD)f(x)$.

This computation shows that, in fact, $Dx=xD+1$.

Review. Linear DEs are those that can be written as $Ly=f(x)$ where L is a linear differential operator: namely,

$$L=p_n(x)D^n+p_{n-1}(x)D^{n-1}+\ldots+p_1(x)D+p_0(x). \tag{1}$$

Recall that the operators xD and Dx are not the same: instead, $Dx=xD+1$.

We say that an operator of the form (1) is in **normal form**.

For instance. xD is in normal form, whereas Dx is not in normal form. It follows from the previous example tha the normal form of Dx is $xD+1$.

Example 42. Let $a=a(x)$ be some function.

(a) Write the operator Da in normal form [normal form means as in (1)].

(b) Write the operator D^2a in normal form.

Solution.

(a) $(Da)f(x)=\frac{\mathrm{d}}{\mathrm{d}x}[a(x)\,f(x)]=a'(x)f(x)+a(x)f'(x)=(a'+aD)f(x)$
Hence, $Da=aD+a'$.

(b) $(D^2a)f(x)=\frac{\mathrm{d}^2}{\mathrm{d}x^2}[a(x)\,f(x)]=\frac{\mathrm{d}}{\mathrm{d}x}[a'(x)f(x)+a(x)f'(x)]=a''(x)f(x)+2a'(x)f'(x)+a(x)f''(x)$
$=(a''+2a'D+aD^2)f(x)$
Hence, $D^2a=aD^2+2a'D+a''$.

Example 43. Suppose that a and b depend on x. Expand $(D+a)(D+b)$ in normal form.

Solution. $(D+a)(D+b) = D^2 + Db + aD + ab = D^2 + (bD + b') + aD + ab = D^2 + (a+b)D + ab + b'$

Comment. Of course, if b is a constant, then $b' = 0$ and we just get the familiar expansion.

Comment. At this point, it is not surprising that, in general, $(D+a)(D+b) \neq (D+b)(D+a)$.

Example 44. Suppose we want to factor $D^2 + pD + q$ as $(D+a)(D+b)$. [p, q, a, b depend on x]

(a) Spell out equations to find a and b.

(b) Find all factorizations of D^2. [An obvious one is $D^2 = D \cdot D$ but there are others!]

Solution.

(a) Matching coefficients with $(D+a)(D+b) = D^2 + (a+b)D + ab + b'$ (we expanded this in the previous example), we find that we need

$$p = a + b, \quad q = ab + b'.$$

Equivalently, $a = p - b$ and $q = (p-b)b + b'$. The latter is a nonlinear (!) DE for b. Once solved for b, we obtain a as $a = p - b$.

(b) This is the case $p = q = 0$. The DE for b becomes $b' = b^2$.
Because it is separable (show all details!), we find that $b(x) = \frac{1}{C - x}$ or $b(x) = 0$.
Since $a = -b$, we obtain the factorizations $D^2 = \left(D - \frac{1}{C-x}\right)\left(D + \frac{1}{C-x}\right)$ and $D^2 = D \cdot D$.
Our computations show that there are no further factorizations.

Comment. Note that this example illustrates that factorization of differential operators is not unique!
For instance, $D^2 = D \cdot D$ and $D^2 = \left(D + \frac{1}{x}\right) \cdot \left(D - \frac{1}{x}\right)$ (the case $C = 0$ above).

Comment. In general, the nonlinear DE for b does not have any polynomial or rational solution (or, in fact, any solution that can be expressed in terms of functions that we are familiar with).

Solving linear recurrences with constant coefficients

Motivation: Fibonacci numbers

The numbers $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$ are called **Fibonacci numbers**.

They are defined by the recursion $F_{n+1} = F_n + F_{n-1}$ and $F_0 = 0$, $F_1 = 1$.

How fast are they growing?
Have a look at ratios of Fibonacci numbers: $\frac{2}{1} = 2$, $\frac{3}{2} = 1.5$, $\frac{5}{3} \approx 1.667$, $\frac{8}{5} = 1.6$, $\frac{13}{8} = 1.625$, $\frac{21}{13} = 1.615$, $\frac{34}{21} = 1.619$, …

These ratios approach the **golden ratio** $\varphi = \frac{1+\sqrt{5}}{2} = 1.618\ldots$

In other words, it appears that $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}$.

We will soon understand where this is coming from.

Armin Straub
straub@southalabama.edu

We can derive all of that using the same ideas as in the case of linear differential equations. The crucial observation that we can write the recursion in operator form:

$F_{n+1} = F_n + F_{n-1}$ is equivalent to $(N^2 - N - 1)F_n = 0$.

Here, N is the shift operator: $Na_n = a_{n+1}$.

Comment. Recurrence equations are discrete analogs of differential equations.

For instance, recall that $f'(x) = \lim_{h\to 0} \frac{1}{h}[f(x+h) - f(x)]$.

Setting $h = 1$, we get the rough estimate $f'(x) \approx f(x+1) - f(x)$ so that D is (roughly) approximated by $N - 1$.

Example 45. Find the general solution to the recursion $a_{n+1} = 7a_n$.

Solution. Note that $a_n = 7a_{n-1} = 7 \cdot 7a_{n-2} = \cdots = 7^n a_0$.

Hence, the general solution is $a_n = C \cdot 7^n$.

Comment. This is analogous to $y' = 7y$ having the general solution $y(x) = Ce^{7x}$.

Solving recurrence equations

Example 46. ("warmup") Find the general solution to the recursion $a_{n+2} = a_{n+1} + 6a_n$.

Solution. The recursion can be written as $p(N)a_n = 0$ where $p(N) = N^2 - N - 6 = (N-3)(N+2)$.

Since $(N-3)a_n = 0$ has solution $a_n = C \cdot 3^n$, and since $(N+2)a_n = 0$ has solution $a_n = C \cdot (-2)^n$ (compare previous example), we conclude that the general solution is $a_n = C_1 \cdot 3^n + C_2 \cdot (-2)^n$.

Comment. This must indeed be the general solution, because the two degrees of freedom C_1, C_2 allow us to match any initial conditions $a_0 = A$, $a_1 = B$: the two equations $C_1 + C_2 = A$ and $3C_1 - 2C_2 = B$ in matrix form are $\begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix}$, which always has a (unique) solution because $\det\left(\begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix}\right) = -5 \neq 0$.

Example 47. (cont'd) Let the sequence a_n be defined by $a_{n+2} = a_{n+1} + 6a_n$ and $a_0 = 1$, $a_1 = 8$.

(a) Determine the first few terms of the sequence.

(b) Find a formula for a_n.

(c) Determine $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

Solution.

(a) $a_2 = a_1 + 6a_0 = 14$, $a_3 = a_2 + 6a_1 = 62$, $a_4 = 146$, ...

(b) The recursion can be written as $p(N)a_n = 0$ where $p(N) = N^2 - N - 6$ has roots $3, -2$.

Hence, $a_n = C_1\, 3^n + C_2\, (-2)^n$ and we only need to figure out the two unknowns C_1, C_2. We can do that using the two initial conditions: $a_0 = C_1 + C_2 = 1$, $a_1 = 3C_1 - 2C_2 = 8$.

Solving, we find $C_1 = 2$ and $C_2 = -1$ so that, in conclusion, $a_n = 2 \cdot 3^n - (-2)^n$.

Comment. Such a formula is sometimes called a **Binet-like formula** (because it is of the same kind as the Binet formula for the Fibonacci numbers that we can derive in the same manner).

(c) It follows from our formula that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 3$ (because $|3| > |-2|$ so that 3^n dominates $(-2)^n$).

To see this, we need to realize that, for large n, 3^n is much larger than $(-2)^n$ so that we have $a_n \approx 2 \cdot 3^n$ when n is large. Hence, $\frac{a_{n+1}}{a_n} \approx \frac{2 \cdot 3^{n+1}}{2 \cdot 3^n} = 3$.

Alternatively, to be very precise, we can observe that (by dividing each term by 3^n)

$$\frac{a_{n+1}}{a_n} = \frac{2 \cdot 3^{n+1} - (-2)^{n+1}}{2 \cdot 3^n - (-2)^n} = \frac{2 \cdot 3 + 2\left(-\frac{2}{3}\right)^n}{2 \cdot 1 - \left(-\frac{2}{3}\right)^n} \xrightarrow{\text{as } n\to\infty} \frac{2 \cdot 3 + 0}{2 \cdot 1 - 0} = 3.$$

Example 48. ("warmup") Find the general solution to the recursion $a_{n+2} = 4a_{n+1} - 4a_n$.

Solution. The recursion can be written as $p(N)a_n = 0$ where $p(N) = N^2 - 4N + 4$ has roots $2, 2$.

So a solution is 2^n and, from our discussion of DEs, it is probably not surprising that a second solution is $n \cdot 2^n$.

Hence, the general solution is $a_n = C_1 \cdot 2^n + C_2 \cdot n \cdot 2^n = (C_1 + C_2 n) \cdot 2^n$.

Comment. This is analogous to $(D-2)^2 y' = 0$ having the general solution $y(x) = (C_1 + C_2 x)e^{2x}$.

Check! Let's check that $a_n = n \cdot 2^n$ indeed satisfies the recursion $(N-2)^2 a_n = 0$.

$(N-2) n \cdot 2^n = (n+1) 2^{n+1} - 2n \cdot 2^n = 2^{n+1}$, so that $(N-2)^2 n \cdot 2^n = (N-2) 2^{n+1} = 0$.

Combined, we obtain the following analog of Theorem 24 for recurrence equations (RE):

Comment. Sequences that are solutions to such recurrences are called **constant recursive** or **C-finite**.

Theorem 49. Consider the homogeneous linear RE with constant coefficients $p(N)a_n = 0$.

- If r is a root of the characteristic polynomial and if k is its multiplicity, then k (independent) solutions of the RE are given by $n^j r^n$ for $j = 0, 1, \ldots, k-1$.
- Combining these solutions for all roots, gives the general solution.

Moreover. If r is the sole largest root by absolute value among the roots contributing to a_n, then $a_n \approx C r^n$ (if r is not repeated—what if it is?) for large n. In particular, it follows that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r.$$

Advanced comment. Things can get weird if there are several roots of the same absolute value. Consider, for instance, the case $a_n = 2^n + (-2)^n$. Can you see that, in this case, the limit $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ doesn't even exist?

Example 50. Find the general solution to the recursion $a_{n+3} = 2a_{n+2} + a_{n+1} - 2a_n$.

Solution. The recursion can be written as $p(N)a_n = 0$ where $p(N) = N^3 - 2N^2 - N + 2$ has roots $2, 1, -1$. (Here, we used some help from a computer algebra system to find the roots.)
Hence, the general solution is $a_n = C_1 \cdot 2^n + C_2 + C_3 \cdot (-1)^n$.

Example 51. Find the general solution to the recursion $a_{n+3} = 3a_{n+2} - 4a_n$.

Solution. The recursion can be written as $p(N)a_n = 0$ where $p(N) = N^3 - 3N^2 + 4$ has roots $2, 2, -1$. (Again, we used some help from a computer algebra system to find the roots.)
Hence, the general solution is $a_n = (C_1 + C_2 n) \cdot 2^n + C_3 \cdot (-1)^n$.

Theorem 52. (Binet's formula) $F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$

Proof. The recursion $F_{n+1} = F_n + F_{n-1}$ can be written as $p(N)a_n = 0$ where $p(N) = N^2 - N - 1$ has roots

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.618, \qquad \lambda_2 = \frac{1-\sqrt{5}}{2} \approx -0.618.$$

Hence, $F_n = C_1 \cdot \lambda_1^n + C_2 \cdot \lambda_2^n$ and we only need to figure out the two unknowns C_1, C_2. We can do that using the two initial conditions: $F_0 = C_1 + C_2 \overset{!}{=} 0$, $F_1 = C_1 \cdot \frac{1+\sqrt{5}}{2} + C_2 \cdot \frac{1-\sqrt{5}}{2} \overset{!}{=} 1$.

Solving, we find $C_1 = \frac{1}{\sqrt{5}}$ and $C_2 = -\frac{1}{\sqrt{5}}$ so that, in conclusion, $F_n = \frac{1}{\sqrt{5}}(\lambda_1^n - \lambda_2^n)$, as claimed. □

Comment. For large n, $F_n \approx \frac{1}{\sqrt{5}}\lambda_1^n$ (because λ_2^n becomes very small). In fact, $F_n = \operatorname{round}\left(\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n\right)$.

Back to the quotient of Fibonacci numbers. In particular, because λ_1^n dominates λ_2^n, it is now transparent that the ratios $\frac{F_{n+1}}{F_n}$ approach $\lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.618$. To be precise, note that

$$\frac{F_{n+1}}{F_n} = \frac{\frac{1}{\sqrt{5}}(\lambda_1^{n+1} - \lambda_2^{n+1})}{\frac{1}{\sqrt{5}}(\lambda_1^n - \lambda_2^n)} = \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1^n - \lambda_2^n} = \frac{\lambda_1 - \lambda_2\left(\frac{\lambda_2}{\lambda_1}\right)^n}{1 - \left(\frac{\lambda_2}{\lambda_1}\right)^n} \overset{n\to\infty}{\longrightarrow} \frac{\lambda_1 - 0}{1 - 0} = \lambda_1.$$

In fact, it follows from $\lambda_2 < 0$ that the ratios $\frac{F_{n+1}}{F_n}$ approach λ_1 in the alternating fashion that we observed numerically earlier. Can you see that?

Example 53. Consider the sequence a_n defined by $a_{n+2} = 4a_{n+1} + 9a_n$ and $a_0 = 1$, $a_1 = 2$. Determine $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

Solution. The recursion can be written as $p(N)a_n = 0$ where $p(N) = N^2 - 4N - 9$ has roots $\frac{4 \pm \sqrt{52}}{2} \approx 5.6056$, -1.6056. Both roots have to be involved in the solution in order to get integer values.
We conclude that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 2 + \sqrt{13} \approx 5.6056$ (because $|5.6056| > |-1.6056|$).

Example 54. (extra) Consider the sequence a_n defined by $a_{n+2} = 2a_{n+1} + 4a_n$ and $a_0 = 0$, $a_1 = 1$. Determine $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

First few terms of sequence. $0, 1, 2, 8, 24, 80, 256, 832, \ldots$
These are actually related to Fibonacci numbers. Indeed, $a_n = 2^{n-1}F_n$. Can you prove this directly from the recursions? Alternatively, this follows from comparing the Binet-like formulas.

Solution. Proceeding as in the previous example, we find $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1 + \sqrt{5} \approx 3.23607$.

Comment. With just a little more work, we find the Binet-like formula $a_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2\sqrt{5}}$.

Sketch of Lecture 10

Example 55. (review) Consider the sequence a_n defined by $a_{n+2} = a_{n+1} + 2a_n$ and $a_0 = 1$, $a_1 = 8$.

(a) Determine the first few terms of the sequence.

(b) Find a formula for a_n.

(c) Determine $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

Solution.

(a) $a_2 = 10$, $a_3 = 26$

(b) The recursion can be written as $p(N)a_n = 0$ where $p(N) = N^2 - N - 2$ has roots $2, -1$.
Hence, $a_n = C_1\, 2^n + C_2\, (-1)^n$ and we only need to figure out the two unknowns C_1, C_2. We can do that using the two initial conditions: $a_0 = C_1 + C_2 = 1$, $a_1 = 2C_1 - C_2 = 8$.
Solving, we find $C_1 = 3$ and $C_2 = -2$ so that, in conclusion, $a_n = 3 \cdot 2^n - 2(-1)^n$.

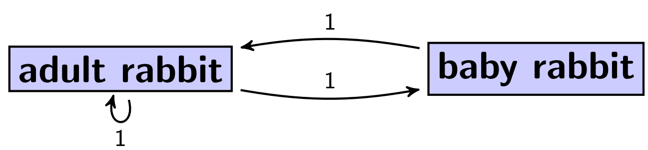
(c) It follows from the formula $a_n = 3 \cdot 2^n - 2(-1)^n$ that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 2$.

Comment. In fact, this already follows from $a_n = C_1\, 2^n + C_2\, (-1)^n$ provided that $C_1 \neq 0$. Since $a_n = C_2\, (-1)^n$ (the case $C_1 = 0$) is not compatible with $a_0 = 1$, $a_1 = 8$, we can conclude $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 2$ without computing the actual values of C_1 and C_2.

Preview: A system of recurrence equations equivalent to the Fibonacci recurrence

Example 56. We model rabbit reproduction as follows.

Each month, every pair of adult rabbits produces one pair of baby rabbit as offspring. Meanwhile, it takes baby rabbits one month to mature to adults.



Comment. In this simplified model, rabbits always come in male/female pairs and no rabbits die. Though these features might make it sound fairly useless, the model may have some merit when describing populations under ideal conditions (unlimited resources) and over short time (no deaths).

Historical comment. The question how many rabbits there are after one year, when starting out with a pair of baby rabbits is famously included in the 1202 textbook of the Italian mathematician Leonardo of Pisa, known as Fibonacci.

If we start with one baby rabbit pair, how many adult rabbits are there after n months?

Solution. Let a_n be the number of adult rabbit pairs after n months. Likewise, b_n is the number of baby rabbit pairs. The transition from one month to the next is given by $a_{n+1} = a_n + b_n$ and $b_{n+1} = a_n$. Using $b_n = a_{n-1}$ (from the second equation) in the first equation, we obtain $a_{n+1} = a_n + a_{n-1}$.

The initial conditions are $a_0 = 0$ and $a_1 = 1$ (the latter follows from $b_0 = 1$).

It follows that the number b_n of adult rabbits are precisely the Fibonacci numbers F_n.

Comment. Note that the transition from one month to the next is described by in matrix-vector form as

$$\begin{bmatrix} a_{n+1} \\ b_{n+1} \end{bmatrix} = \begin{bmatrix} a_n + b_n \\ a_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_n \\ b_n \end{bmatrix}.$$

Writing $\boldsymbol{a}_n = \begin{bmatrix} a_n \\ b_n \end{bmatrix}$, this becomes $\boldsymbol{a}_{n+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \boldsymbol{a}_n$ with $\boldsymbol{a}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Consequently, $\boldsymbol{a}_n = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \boldsymbol{a}_0 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Armin Straub
straub@southalabama.edu

Looking ahead. Can you see how, starting with the Fibonacci recurrence $F_{n+2} = F_{n+1} + F_n$, we can arrive at this same system?

Solution. Set $\boldsymbol{a}_n = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$. Then $\boldsymbol{a}_{n+1} = \begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} F_{n+1} + F_n \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \boldsymbol{a}_n$.

Crash course: Eigenvalues and eigenvectors

If $A\boldsymbol{x} = \lambda \boldsymbol{x}$ (and $\boldsymbol{x} \neq \boldsymbol{0}$), then $\boldsymbol{x}$ is an **eigenvector** of A with **eigenvalue** λ (just a number).

Note that for the equation $A\boldsymbol{x} = \lambda \boldsymbol{x}$ to make sense, A needs to be a square matrix (i.e. $n \times n$).

Key observation:

$$\begin{aligned} & A\boldsymbol{x} = \lambda \boldsymbol{x} \\ \Longleftrightarrow\ & A\boldsymbol{x} - \lambda \boldsymbol{x} = \boldsymbol{0} \\ \Longleftrightarrow\ & (A - \lambda I)\boldsymbol{x} = \boldsymbol{0} \end{aligned}$$

This homogeneous system has a nontrivial solution $\boldsymbol{x}$ if and only if $\det(A - \lambda I) = 0$.

To find eigenvectors and eigenvalues of A:

(a) First, find the eigenvalues λ by solving $\det(A - \lambda I) = 0$.

$\det(A - \lambda I)$ is a polynomial in λ, called the **characteristic polynomial** of A.

(b) Then, for each eigenvalue λ, find corresponding eigenvectors by solving $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$.

Example 57. Determine the eigenvalues and eigenvectors of $A = \begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix}$.

Solution. The characteristic polynomial is:

$\det(A - \lambda I) = \det\left(\begin{bmatrix} 8-\lambda & -10 \\ 5 & -7-\lambda \end{bmatrix} \right) = (8-\lambda)(-7-\lambda) + 50 = \lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2)$

Hence, the eigenvalues are $\lambda = 3$ and $\lambda = -2$.

- To find an eigenvector for $\lambda = 3$, we need to solve $\begin{bmatrix} 5 & -10 \\ 5 & -10 \end{bmatrix} \boldsymbol{x} = \boldsymbol{0}$.
  Hence, $\boldsymbol{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 3$.
- To find an eigenvector for $\lambda = -2$, we need to solve $\begin{bmatrix} 10 & -10 \\ 5 & -5 \end{bmatrix} \boldsymbol{x} = \boldsymbol{0}$.
  Hence, $\boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = -2$.

Check! $\begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix} = -2 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

On the other hand, a random other vector like $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is not an eigenvector: $\begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -12 \\ -9 \end{bmatrix} \neq \lambda \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Example 58. (homework) Determine the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & -6 \\ 1 & -4 \end{bmatrix}$.

Solution. (final answer only) $\boldsymbol{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = -2$, and $\boldsymbol{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = -1$.

Example 59. (review) Consider the sequence a_n defined by $a_{n+2} = 4a_n - 3a_{n+1}$ and $a_0 = 1$, $a_1 = 2$. Determine $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

Solution. The recursion can be written as $p(N)a_n = 0$ where $p(N) = N^2 + 3N - 4$ has roots $1, -4$. Hence, the general solution is $a_n = C_1 + C_2 \cdot (-4)^n$. We can see that both roots have to be involved in the solution (in other words, $C_1 \neq 0$ and $C_2 \neq 0$) because $a_n = C_1$ and $a_n = C_2 \cdot (-4)^n$ are not consistent with the initial conditions. We conclude that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = -4$ (because $|-4| > |1|$).

Systems of recurrence equations

Example 60. Write the (second-order) RE $a_{n+2} = 4a_n - 3a_{n+1}$, with $a_0 = 1$, $a_1 = 2$, as a system of (first-order) recurrences.

Solution. (long form) Write $b_n = a_{n+1}$.

Then, $a_{n+2} = 4a_n - 3a_{n+1}$ translates into the first-order system $\begin{cases} a_{n+1} = b_n \\ b_{n+1} = 4a_n - 3b_n \end{cases}$.

Let $\boldsymbol{a}_n = \begin{bmatrix} a_n \\ b_n \end{bmatrix}$. Then, in matrix form, the RE is $\boldsymbol{a}_{n+1} = \begin{bmatrix} 0 & 1 \\ 4 & -3 \end{bmatrix} \boldsymbol{a}_n$, with $\boldsymbol{a}_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Solution. (short form) Equivalently, write $\boldsymbol{a}_n = \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix}$. Then we obtain the system

$$\boldsymbol{a}_{n+1} = \begin{bmatrix} a_{n+1} \\ a_{n+2} \end{bmatrix} = \begin{bmatrix} a_{n+1} \\ 4a_n - 3a_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 4 & -3 \end{bmatrix} \boldsymbol{a}_n, \quad \boldsymbol{a}_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Comment. It follows that $\boldsymbol{a}_n = \begin{bmatrix} 0 & 1 \\ 4 & -3 \end{bmatrix}^n \boldsymbol{a}_0 = \begin{bmatrix} 0 & 1 \\ 4 & -3 \end{bmatrix}^n \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Solving (systems of) REs is equivalent to computing powers of matrices!

Comment. We could also write $\boldsymbol{a}_n = \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$ (with the order of the entries reversed). In that case, the system is

$$\boldsymbol{a}_{n+1} = \begin{bmatrix} a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 4a_n - 3a_{n+1} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 1 & 0 \end{bmatrix} \boldsymbol{a}_n, \quad \boldsymbol{a}_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Comment. Recall that the **characteristic polynomial** of a matrix M is $\det(M - \lambda I)$. Compute the characteristic polynomial of both $M = \begin{bmatrix} 0 & 1 \\ 4 & -3 \end{bmatrix}$ and $M = \begin{bmatrix} -3 & 4 \\ 1 & 0 \end{bmatrix}$. In both cases, we get $\lambda^2 + 3\lambda - 4$, which matches the polynomial $p(N)$ (also called characteristic polynomial!) in the previous example. This will always happen and explains why both are referred to as the characteristic polynomial.

Example 61. Write $a_{n+3} - 4a_{n+2} + a_{n+1} + 6a_n = 0$ as a system of (first-order) recurrences.

Solution. Write $\boldsymbol{a}_n = \begin{bmatrix} a_n \\ a_{n+1} \\ a_{n+2} \end{bmatrix}$. Then we obtain the system

$$\boldsymbol{a}_{n+1} = \begin{bmatrix} a_{n+1} \\ a_{n+2} \\ a_{n+3} \end{bmatrix} = \begin{bmatrix} a_{n+1} \\ a_{n+2} \\ 4a_{n+2} - a_{n+1} - 6a_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -1 & 4 \end{bmatrix} \begin{bmatrix} a_n \\ a_{n+1} \\ a_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -1 & 4 \end{bmatrix} \boldsymbol{a}_n.$$

In summary, the RE in matrix form is $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$ with M the matrix above.

Important comment. Consequently, $\boldsymbol{a}_n = M^n \boldsymbol{a}_0$.

Armin Straub
straub@southalabama.edu

Solving systems of recurrence equations

(systems of REs) The unique solution to $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$, $\boldsymbol{a}_0 = \boldsymbol{c}$ is $\boldsymbol{a}_n = M^n \boldsymbol{c}$.

We will say that M^n is the **fundamental matrix solution** to $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$ with $\boldsymbol{a}_0 = I$ (the identity matrix).

Comment. Note that $M^n\boldsymbol{c}$ is a linear combination of the columns of M^n. In other words, each column of M^n is a solution to $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$ and $M^n\boldsymbol{c}$ is the general solution.

In general, we say that a matrix Φ_n is a **matrix solution** to $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$ if $\Phi_{n+1} = M\Phi_n$. Φ_n being a matrix solution is equivalent to each column of Φ_n being a normal (vector) solution. If the general solution of $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$ can be obtained as the linear combination of the columns of Φ_n, then Φ_n is a **fundamental matrix solution**. Above, we observed that $\Phi_n = M^n$ is a special fundamental matrix solution.

If this is a bit too abstract at this point, have a look at Example 62 below.

However, at this point, it remains to figure out how to actually compute M^n. We will actually proceed the other way around: we construct the general solution of $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$ (see the box below) and then we use that to determine M^n from that.

To solve $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$, determine the eigenvectors of M.

- Each λ-eigenvector $\boldsymbol{v}$ provides a solution: $\boldsymbol{a}_n = \boldsymbol{v}\lambda^n$ [assuming that $\lambda \neq 0$]
- If there are enough eigenvectors, these combine to the general solution.

Why? If $\boldsymbol{a}_n = \boldsymbol{v}\lambda^n$ for a λ-eigenvector $\boldsymbol{v}$, then $\boldsymbol{a}_{n+1} = \boldsymbol{v}\lambda^{n+1}$ and $M\boldsymbol{a}_n = M\boldsymbol{v}\lambda^n = \lambda\boldsymbol{v}\cdot\lambda^n = \boldsymbol{v}\lambda^{n+1}$.

Where is this coming from? When solving single linear recurrences, we found that the basic solutions are of the form cr^n where $r \neq 0$ is a root of the characteristic polynomials. To solve $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$, it is therefore natural to look for solutions of the form $\boldsymbol{a}_n = \boldsymbol{c}r^n$ (where $\boldsymbol{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$). Note that $\boldsymbol{a}_{n+1} = \boldsymbol{c}r^{n+1} = r\boldsymbol{a}_n$.

Plugging into $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$ we find $\boldsymbol{c}r^{n+1} = M\boldsymbol{c}r^n$.

Cancelling r^n (just a nonzero number!), this simplifies to $r\boldsymbol{c} = M\boldsymbol{c}$.

In other words, $\boldsymbol{a}_n = \boldsymbol{c}r^n$ is a solution if and only if $\boldsymbol{c}$ is an r-eigenvector of M.

Comment. If there are not enough eigenvectors, then we know what to do as well (at least in principle): instead of looking only for solutions of the type $\boldsymbol{a}_n = \boldsymbol{v}\lambda^n$, we also need to look for solutions of the type $\boldsymbol{a}_n = (\boldsymbol{v}n + \boldsymbol{w})\lambda^n$. Note that this can only happen if an eigenvalue is a repeated root of the characteristic polynomial.

Example 62. Let $M = \begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix}$.

(a) Determine the general solution to $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$.

(b) Determine a **fundamental matrix solution** to $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$.

(c) Compute M^n.

Solution.

(a) Recall that each λ-eigenvector $\boldsymbol{v}$ of M provides us with a solution: $\boldsymbol{a}_n = \boldsymbol{v}\lambda^n$

We computed in Example 57 that $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 3$, and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = -2$.

Hence, the general solution is $C_1\begin{bmatrix} 2 \\ 1 \end{bmatrix}3^n + C_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}(-2)^n$.

(b) Note that we can write the general solution as

$\boldsymbol{a}_n = C_1\begin{bmatrix} 2 \\ 1 \end{bmatrix}3^n + C_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}(-2)^n = \begin{bmatrix} 2\cdot 3^n & (-2)^n \\ 3^n & (-2)^n \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$.

We call $\Phi_n = \begin{bmatrix} 2\cdot 3^n & (-2)^n \\ 3^n & (-2)^n \end{bmatrix}$ the corresponding **fundamental matrix (solution)**.

Note that our general solution is precisely $\Phi_n \boldsymbol{c}$ with $\boldsymbol{c} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$.

Observations.

(a) The columns of Φ_n are (independent) solutions of the system.

(b) Φ_n solves the RE itself: $\Phi_{n+1} = M\Phi_n$.
[Spell this out in this example! That Φ_n solves the RE follows from the definition of matrix multiplication.]

(c) It follows that $\Phi_n = M^n\Phi_0$. Equivalently, $\Phi_n\Phi_0^{-1} = M^n$. (See next part!)

(c) Note that $\Phi_0 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, so that $\Phi_0^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$. It follows that

$$M^n = \Phi_n\Phi_0^{-1} = \begin{bmatrix} 2\cdot 3^n & (-2)^n \\ 3^n & (-2)^n \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2\cdot 3^n - (-2)^n & -2\cdot 3^n + 2(-2)^n \\ 3^n - (-2)^n & -3^n + 2(-2)^n \end{bmatrix}.$$

Check. Let us verify the formula for M^n in the cases $n = 0$ and $n = 1$:

$M^0 = \begin{bmatrix} 2-1 & -2+2 \\ 1-1 & -1+2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$M^1 = \begin{bmatrix} 2\cdot 3-(-2) & -2\cdot 3+2(-2) \\ 3-(-2) & -3+2(-2) \end{bmatrix} = \begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix}$

We just saw that being able to compute matrix powers is equivalent to solving systems of recurrences. Indeed, we can use this fact to compute matrix powers.

(a way to compute powers of a matrix M)
Compute a fundamental matrix solution Φ_n of $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$.
Then $M^n = \Phi_n\Phi_0^{-1}$.

If you have taken linear algebra classes, you may have learned that matrix powers M^n can be computed by diagonalizing the matrix M. The latter hinges on computing eigenvalues and eigenvectors of M as well. Compare the two approaches!

(systems of REs) The unique solution to $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$, $\boldsymbol{a}_0 = \boldsymbol{c}$ is $\boldsymbol{a}_n = M^n\boldsymbol{c}$.

- Here, M^n is the fundamental matrix solution to $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$, $\boldsymbol{a}_0 = I$ (with I the identity matrix).
- If Φ_n is any fundamental matrix solution to $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$, then $M^n = \Phi_n\Phi_0^{-1}$.
- To construct a fundamental matrix solution Φ_n, we compute eigenvectors:
  Given a λ-eigenvector $\boldsymbol{v}$, we have the corresponding solution $\boldsymbol{a}_n = \boldsymbol{v}\lambda^n$.
  If there are enough eigenvectors, we can collect these as columns to obtain Φ_n.

Why? Since Φ_n is a fundamental matrix solution, we have $\Phi_{n+1} = M\Phi_n$ and, thus, $\Phi_n = M^n\Phi_0$.
It follows that $M^n = \Phi_n\Phi_0^{-1}$.

Example 63. (review) Write the (second-order) RE $a_{n+2} = a_{n+1} + 2a_n$, with $a_0 = 0$, $a_1 = 1$, as a system of (first-order) recurrences.

Solution. If $\boldsymbol{a}_n = \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix}$, then $\boldsymbol{a}_{n+1} = \begin{bmatrix} a_{n+1} \\ a_{n+2} \end{bmatrix} = \begin{bmatrix} a_{n+1} \\ a_{n+1} + 2a_n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}\boldsymbol{a}_n$ with $\boldsymbol{a}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Example 64. Let $M = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$.

(a) Determine the general solution to $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$.

(b) Determine a fundamental matrix solution to $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$.

(c) Compute M^n.

(d) Solve $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$, $\boldsymbol{a}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Solution.

(a) Recall that each λ-eigenvector $\boldsymbol{v}$ of M provides us with a solution: namely, $\boldsymbol{a}_n = \boldsymbol{v}\lambda^n$.

The characteristic polynomial is: $\det(A - \lambda I) = \det\left(\begin{bmatrix} -\lambda & 1 \\ 2 & 1-\lambda \end{bmatrix}\right) = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)$.

Hence, the eigenvalues are $\lambda = 2$ and $\lambda = -1$.

- $\lambda = 2$: Solving $\begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix}\boldsymbol{v} = \boldsymbol{0}$, we find that $\boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector for $\lambda = 2$.
- $\lambda = -1$: Solving $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\boldsymbol{v} = \boldsymbol{0}$, we find that $\boldsymbol{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = -1$.

Hence, the general solution is $C_1\begin{bmatrix} 1 \\ 2 \end{bmatrix}2^n + C_2\begin{bmatrix} -1 \\ 1 \end{bmatrix}(-1)^n$.

(b) Note that $C_1\begin{bmatrix} 1 \\ 2 \end{bmatrix}2^n + C_2\begin{bmatrix} -1 \\ 1 \end{bmatrix}(-1)^n = \begin{bmatrix} 2^n & -(-1)^n \\ 2\cdot 2^n & (-1)^n \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$.

Hence, a fundamental matrix solution is $\Phi_n = \begin{bmatrix} 2^n & -(-1)^n \\ 2\cdot 2^n & (-1)^n \end{bmatrix}$.

Comment. Other choices are possible and natural. For instance, the order of the two columns is based on our choice of starting with $\lambda = 2$. Also, the columns can be scaled by any constant (for instance, using $-\boldsymbol{v}$ instead of $\boldsymbol{v}$ for $\lambda = -1$ above, we end up with the same Φ_n but with the second column scaled by -1). In general, if Φ_n is a fundamental matrix solution, then so is $\Phi_n C$ where C is an invertible 2×2 matrix.

(c) We compute $M^n = \Phi_n\Phi_0^{-1}$ using $\Phi_n = \begin{bmatrix} 2^n & -(-1)^n \\ 2\cdot 2^n & (-1)^n \end{bmatrix}$. Since $\Phi_0^{-1} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$, we have

$$M^n = \Phi_n\Phi_0^{-1} = \begin{bmatrix} 2^n & -(-1)^n \\ 2\cdot 2^n & (-1)^n \end{bmatrix}\frac{1}{3}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2^n + 2(-1)^n & 2^n - (-1)^n \\ 2\cdot 2^n - 2(-1)^n & 2\cdot 2^n + (-1)^n \end{bmatrix}.$$

(d) $\boldsymbol{a}_n = M^n\boldsymbol{a}_0 = \frac{1}{3}\begin{bmatrix} 2^n + 2(-1)^n & 2^n - (-1)^n \\ 2\cdot 2^n - 2(-1)^n & 2\cdot 2^n + (-1)^n \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2^n - (-1)^n \\ 2\cdot 2^n + (-1)^n \end{bmatrix}$

Alternative solution of the first part. We saw in Example 63 that this system can be obtained from $a_{n+2} = a_{n+1} + 2a_n$ if we set $\boldsymbol{a} = \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix}$. In Example 55, we found that this RE has solutions $a_n = 2^n$ and $a_n = (-1)^n$. Correspondingly, $\boldsymbol{a}_{n+1} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \boldsymbol{a}_n$ has solutions $\boldsymbol{a}_n = \begin{bmatrix} 2^n \\ 2^{n+1} \end{bmatrix}$ and $\boldsymbol{a}_n = \begin{bmatrix} (-1)^n \\ (-1)^{n+1} \end{bmatrix}$.

These combine to the general solution $C_1 \begin{bmatrix} 2^n \\ 2^{n+1} \end{bmatrix} + C_2 \begin{bmatrix} (-1)^n \\ (-1)^{n+1} \end{bmatrix}$ (equivalent to our solution above).

Alternative for last part. Solve the RE from Example 63 to find $a_n = \frac{1}{3}(2^n - (-1)^n)$. The above is $\boldsymbol{a}_n = \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix}$.

Sage. Once we are comfortable with these computations, we can let Sage do them for us.

```
>>> M = matrix([[0,1],[2,1]])
>>> M^2
```

$$\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$$

Verify that this matrix matches what our formula for M^n produces for $n = 2$. In order to reproduce the general formula for M^n, we need to first define n as a symbolic variable:

```
>>> n = var('n')
>>> M^n
```

$$\begin{pmatrix} \frac{1}{3} \cdot 2^n + \frac{2}{3}(-1)^n & \frac{1}{3} \cdot 2^n - \frac{1}{3}(-1)^n \\ \frac{2}{3} \cdot 2^n - \frac{2}{3}(-1)^n & \frac{2}{3} \cdot 2^n + \frac{1}{3}(-1)^n \end{pmatrix}$$

Note that this indeed matches our earlier formula. Can you see how we can read off the eigenvalues and eigenvectors of M from this formula for M^n? Of course, Sage can readily compute these for us directly using, for instance, `M.eigenvectors_right()`. Try it! Can you interpret the output?

Preview: The corresponding system of differential equations

Example 65. Write the (second-order) initial value problem $y'' = y' + 2y$, $y(0) = 0$, $y'(0) = 1$ as a first-order system.

Solution. If $\boldsymbol{y} = \begin{bmatrix} y \\ y' \end{bmatrix}$, then $\boldsymbol{y}' = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} y' \\ y' + 2y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \boldsymbol{y}$ with $\boldsymbol{y}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Homework. Solve this IVP to find $y(x) = \frac{1}{3}(e^{2x} - e^{-x})$. Then compare with the next example.

Example 66. Let $M = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$.

(a) Determine the general solution to $\boldsymbol{y}' = M\boldsymbol{y}$.

(b) Determine a fundamental matrix solution to $\boldsymbol{y}' = M\boldsymbol{y}$.

(c) Solve $\boldsymbol{y}' = M\boldsymbol{y}$, $\boldsymbol{y}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Solution. In Example 63, we only need to replace 2^n by e^{2x} (root 2) and $(-1)^n$ by e^{-x} (root -1)!

(a) The general solution is $C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2x} + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-x}$.

(b) A fundamental matrix solution is $\Phi(x) = \begin{bmatrix} e^{2x} & -e^{-x} \\ 2 \cdot e^{2x} & e^{-x} \end{bmatrix}$.

(c) $\boldsymbol{y}(x) = \frac{1}{3} \begin{bmatrix} e^{2x} - e^{-x} \\ 2 \cdot e^{2x} + e^{-x} \end{bmatrix}$

Preview. The special fundamental matrix M^n will be replaced by e^{Mx}, the **matrix exponential**.

Armin Straub
straub@southalabama.edu

Example 67. (homework)

(a) Write the recurrence $a_{n+3}-4a_{n+2}+a_{n+1}+6a_n=0$ as a system $\boldsymbol{a}_{n+1}=M\boldsymbol{a}_n$ of (first-order) recurrences.

(b) Determine a fundamental matrix solution to $\boldsymbol{a}_{n+1}=M\boldsymbol{a}_n$.

(c) Compute M^n.

Solution.

(a) If $\boldsymbol{a}_n=\begin{bmatrix} a_n \\ a_{n+1} \\ a_{n+2} \end{bmatrix}$, then the RE becomes $\boldsymbol{a}_{n+1}=M\boldsymbol{a}_n$ with $M=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -1 & 4 \end{bmatrix}$.

(b) Because we started with a single (third-order) equation, we can avoid computing eigenvectors and eigenvalues (indeed, we will find these as a byproduct).

By factoring the characteristic equation $N^3-4N^2+N+6=(N-3)(N-2)(N+1)$, we find that the characteristic roots are $3,2,-1$ (these are also precisely the eigenvalues of M).

Hence, $a_n=C_1\cdot 3^n+C_2\cdot 2^n+C_3\cdot(-1)^n$ is the general solution to the initial RE.

Correspondingly, a fundamental matrix solution of the system is $\Phi_n=\begin{bmatrix} 3^n & 2^n & (-1)^n \\ 3\cdot 3^n & 2\cdot 2^n & -(-1)^n \\ 9\cdot 3^n & 4\cdot 2^n & (-1)^n \end{bmatrix}$.

Note. This tells us that $\begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$ is a 3-eigenvector, $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ a 2-eigenvector, and $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ a -1-eigenvector of M.

(c) Since $\Phi_{n+1}=M\Phi_n$, we have $\Phi_n=M^n\Phi_0$ so that $M^n=\Phi_n\Phi_0^{-1}$. This allows us to compute that:

$$M^n=\frac{1}{12}\begin{bmatrix} -6\cdot 3^n+12\cdot 2^n+6(-1)^n & -3\cdot 3^n+8\cdot 2^n-5(-1)^n & 3\cdot 3^n-4\cdot 2^n+(-1)^n \\ -18\cdot 3^n+24\cdot 2^n-6(-1)^n & \ldots & \ldots \\ -54\cdot 3^n+48\cdot 2^n+6(-1)^n & \ldots & \ldots \end{bmatrix}$$

Example 68. (homework) If $M=\begin{bmatrix} 3 & & & \\ & -2 & & \\ & & 5 & \\ & & & 1 \end{bmatrix}$, what is M^n?

Comment. Entries that are not printed are meant to be zero (to make the structure of the 4×4 matrix more visibly transparent).

Solution. $M^n=\begin{bmatrix} 3^n & & & \\ & (-2)^n & & \\ & & 5^n & \\ & & & 1 \end{bmatrix}$

If this isn't clear to you, multiply out M^2. What happens?

Armin Straub
straub@southalabama.edu

Systems of differential equations

Example 69. (review) Write the (second-order) differential equation $y'' = 2y' + y$ as a system of (first-order) differential equations.

Solution. If $\boldsymbol{y} = \begin{bmatrix} y \\ y' \end{bmatrix}$, then $\boldsymbol{y}' = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} y' \\ 2y' + y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \boldsymbol{y}$. For short, $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \boldsymbol{y}$.

Comment. Hence, we care about systems of differential equations, even if we work with just one function.

Example 70. Write the (third-order) differential equation $y''' = 3y'' - 2y' + y$ as a system of (first-order) differential equations.

Solution. If $\boldsymbol{y} = \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix}$, then $\boldsymbol{y}' = \begin{bmatrix} y' \\ y'' \\ y''' \end{bmatrix} = \begin{bmatrix} y' \\ y'' \\ 3y'' - 2y' + y \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -2 & 3 \end{bmatrix} \boldsymbol{y}$.

Example 71. Consider the following system of (second-order) initial value problems:

$$\begin{aligned} y_1'' &= 2y_1' - 3y_2' + 7y_2 \\ y_2'' &= 4y_1' + y_2' - 5y_1 \end{aligned} \qquad y_1(0) = 2, \;\; y_1'(0) = 3, \;\; y_2(0) = -1, \;\; y_2'(0) = 1$$

Write it as a first-order initial value problem in the form $\boldsymbol{y}' = M\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$.

Solution. If $\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_1' \\ y_2' \end{bmatrix}$, then $\boldsymbol{y}' = \begin{bmatrix} y_1' \\ y_2' \\ y_1'' \\ y_2'' \end{bmatrix} = \begin{bmatrix} y_1' \\ y_2' \\ 2y_1' - 3y_2' + 7y_2 \\ 4y_1' + y_2' - 5y_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 7 & 2 & -3 \\ -5 & 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 7 & 2 & -3 \\ -5 & 0 & 4 & 1 \end{bmatrix} \boldsymbol{y}$.

For short, the system translates into $\boldsymbol{y}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 7 & 2 & -3 \\ -5 & 0 & 4 & 1 \end{bmatrix} \boldsymbol{y}$ with $\boldsymbol{y}(0) = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \end{bmatrix}$.

To solve $\boldsymbol{y}' = M\boldsymbol{y}$, determine the eigenvectors of M.

- Each λ-eigenvector $\boldsymbol{v}$ provides a solution: $\boldsymbol{y}(x) = \boldsymbol{v}e^{\lambda x}$
- If there are enough eigenvectors, these combine to the general solution.

(systems of DEs) The unique solution to $\boldsymbol{y}' = M\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{c}$ is $\boldsymbol{y}(x) = e^{Mx}\boldsymbol{c}$.

- Here, e^{Mx} is the fundamental matrix solution to $\boldsymbol{y}' = M\boldsymbol{y}$, $\boldsymbol{y}(0) = I$ (with I the identity matrix).
- If $\Phi(x)$ is any fundamental matrix solution to $\boldsymbol{y}' = M\boldsymbol{y}$, then $e^{Mx} = \Phi(x)\Phi(0)^{-1}$.
- To construct a fundamental matrix solution $\Phi(x)$, we compute eigenvectors:
  Given a λ-eigenvector $\boldsymbol{v}$, we have the corresponding solution $\boldsymbol{y}(x) = \boldsymbol{v}e^{\lambda x}$.
  If there are enough eigenvectors, we can collect these as columns to obtain $\Phi(x)$.

Note. We are defining the **matrix exponential** e^{Mx} as the solution to an IVP. This is equivalent to how one can define the ordinary exponential e^x as the solution to $y' = y$, $y(0) = 1$.
[In a little bit, we will also discuss how to think about the matrix exponential e^{Mx} using power series.]

Comment. If there are not enough eigenvectors, then we know what to do (at least in principle): instead of looking only for solutions of the type $\boldsymbol{y}(x) = \boldsymbol{v}e^{\lambda x}$, we also need to look for solutions of the type $\boldsymbol{y}(x) = (\boldsymbol{v}x + \boldsymbol{w})e^{\lambda x}$. Note that this can only happen if an eigenvalue is a repeated root of the characteristic polynomial.

Armin Straub
straub@southalabama.edu

Important. Compare this to our method of solving systems of REs and for computing matrix powers M^n. Note that the above conclusion about systems of DEs can be deduced along the same lines as what we did for REs:

- If $\Phi(x)$ is a fundamental matrix solution, then so is $\Psi(x) = \Phi(x)C$ for every constant matrix C. (Why?!)
  Therefore, $\Psi(x) = \Phi(x)\Phi(0)^{-1}$ is a fundamental matrix solution with $\Psi(0) = \Phi(0)\Phi(0)^{-1} = I$.
  But e^{Mx} is defined to be the unique such solution, so that $\Psi(x) = e^{Mx}$.

- Let us look for solutions of $\boldsymbol{y}' = M\boldsymbol{y}$ of the form $\boldsymbol{y}(x) = \boldsymbol{v}e^{\lambda x}$. Note that $\boldsymbol{y}' = \lambda\boldsymbol{v}e^{\lambda x} = \lambda\boldsymbol{y}$.
  Plugging into $\boldsymbol{y}' = M\boldsymbol{y}$, we find $\lambda\boldsymbol{y} = M\boldsymbol{y}$.
  In other words, $\boldsymbol{y}(x) = \boldsymbol{v}e^{\lambda x}$ is a solution if and only if $\boldsymbol{v}$ is a λ-eigenvector of M.

Observe how the next example proceeds along the same lines as Example 62.

Important. In fact, we can translate back and forth (without additional computations) by simply replacing 3^n and $(-2)^n$ by e^{3x} and e^{-2x}.

Example 72. (homework) Let $M = \begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix}$.

(a) Determine the general solution to $\boldsymbol{y}' = M\boldsymbol{y}$.

(b) Determine a fundamental matrix solution to $\boldsymbol{y}' = M\boldsymbol{y}$.

(c) Compute e^{Mx}.

(d) Solve the initial value problem $\boldsymbol{y}' = M\boldsymbol{y}$ with $\boldsymbol{y}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Solution.

(a) Recall that each λ-eigenvector $\boldsymbol{v}$ of M provides us with a solution: namely, $\boldsymbol{y}(x) = \boldsymbol{v}e^{\lambda x}$.
We computed earlier that $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 3$, and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = -2$.
Hence, the general solution is $C_1\begin{bmatrix} 2 \\ 1 \end{bmatrix}e^{3x} + C_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{-2x}$.

(b) The corresponding fundamental matrix solution is $\Phi(x) = \begin{bmatrix} 2 \cdot e^{3x} & e^{-2x} \\ e^{3x} & e^{-2x} \end{bmatrix}$.
[Note that our general solution is precisely $\Phi(x)\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$.]

(c) Since $\Phi(0) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, we have $\Phi(0)^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$. It follows that

$$e^{Mx} = \Phi(x)\Phi(0)^{-1} = \begin{bmatrix} 2 \cdot e^{3x} & e^{-2x} \\ e^{3x} & e^{-2x} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 \cdot e^{3x} - e^{-2x} & -2 \cdot e^{3x} + 2e^{-2x} \\ e^{3x} - e^{-2x} & -e^{3x} + 2e^{-2x} \end{bmatrix}.$$

Check. Let us verify the formula for e^{Mx} in the simple case $x = 0$: $e^{M0} = \begin{bmatrix} 2-1 & -2+2 \\ 1-1 & -1+2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(d) The solution to the IVP is $\boldsymbol{y}(x) = e^{Mx}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \cdot e^{3x} + 2e^{-2x} \\ -e^{3x} + 2e^{-2x} \end{bmatrix}$ (the second column of e^{Mx}).

Sage. We can compute the matrix exponential in Sage as follows:

```
>>> M = matrix([[8,-10],[5,-7]])
>>> exp(M*x)
```

$$\begin{pmatrix} (2\,e^{(5\,x)} - 1)\,e^{(-2\,x)} & -2\,(e^{(5\,x)} - 1)\,e^{(-2\,x)} \\ (e^{(5\,x)} - 1)\,e^{(-2\,x)} & -(e^{(5\,x)} - 2)\,e^{(-2\,x)} \end{pmatrix}$$

Note that this indeed matches the result of our computation.
[By the way, the variable x is pre-defined as a symbolic variable in Sage. That's why, unlike for n in the computation of M^n, we did not need to use `x = var('x')` first.]

Example 73. Suppose that $e^{Mx} = \frac{1}{10}\begin{bmatrix} e^x + 9e^{2x} & 3e^x - 3e^{2x} \\ 3e^x - 3e^{2x} & 9e^x + e^{2x} \end{bmatrix}$.

(a) Without doing any computations, determine M^n.

(b) What is M?

(c) Without doing any computations, determine the eigenvalues and eigenvectors of M.

(d) From these eigenvalues and eigenvectors, write down a simple fundamental matrix solution to $\boldsymbol{y}' = M\boldsymbol{y}$.

(e) From that fundamental matrix solution, how can we compute e^{Mx}? (If we didn't know it already...)

(f) Having computed e^{Mx}, what is a simple check that we can (should!) make?

Solution.

(a) Since e^x and e^{2x} correspond to eigenvalues 1 and 2, we just need to replace these by $1^n = 1$ and 2^n:

$$M^n = \frac{1}{10}\begin{bmatrix} 1 + 9 \cdot 2^n & 3 - 3 \cdot 2^n \\ 3 - 3 \cdot 2^n & 9 + 2^n \end{bmatrix}$$

(b) We can simply set $n = 1$ in our formula for M^n, to get $M = \frac{1}{10}\begin{bmatrix} 19 & -3 \\ -3 & 11 \end{bmatrix}$.

(c) The eigenvalues are 1 and 2 (because e^{Mx} contains the exponentials e^x and e^{2x}).
Looking at the coefficients of e^x in the first column of e^{Mx}, we see that $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is a 1-eigenvector.
[We can also look the second column of e^{Mx}, to obtain $\begin{bmatrix} 3 \\ 9 \end{bmatrix}$ which is a multiple and thus equivalent.]
Likewise, we find that $\begin{bmatrix} 9 \\ -3 \end{bmatrix}$ or, equivalently, $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$ is a 2-eigenvector.

Comment. For the above to make sense, we need to keep in mind that, associated to a λ-eigenvector $\boldsymbol{v}$, we have the corresponding solution $\boldsymbol{y}(x) = \boldsymbol{v}e^{\lambda x}$ of the DE $\boldsymbol{y}' = M\boldsymbol{y}$. On the other hand, the columns of e^{Mx} are solutions to that DE and, therefore, must be linear combinations of these $\boldsymbol{v}e^{\lambda x}$.

(d) From the eigenvalues and eigenvectors, we know that $\begin{bmatrix} 1 \\ 3 \end{bmatrix}e^x$ and $\begin{bmatrix} -3 \\ 1 \end{bmatrix}e^{2x}$ are solutions (and that the general solutions consists of the linear combinations of these two).
Selecting these as the columns, we obtain the fundamental matrix solution $\Phi(x) = \begin{bmatrix} e^x & -3e^{2x} \\ 3e^x & e^{2x} \end{bmatrix}$.

Comment. The *fundamental* refers to the fact that the columns combine to the general solution.
The *matrix solution* means that $\Phi(x)$ itself satisfies the DE: namely, we have $\Phi' = M\Phi$. That this is the case is a consequence of matrix multiplication (namely, say, the second column of $M\Phi$ is defined to be M times the second column of Φ; but that column is a vector solution and therefore solves the DE).

(e) We can compute e^{Mx} as $e^{Mx} = \Phi(x)\Phi(0)^{-1}$.
If $\Phi(x) = \begin{bmatrix} e^x & -3e^{2x} \\ 3e^x & e^{2x} \end{bmatrix}$, then $\Phi(0) = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}$ and, hence, $\Phi(0)^{-1} = \frac{1}{10}\begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}$. It follows that

$$e^{Mx} = \Phi(x)\Phi(0)^{-1} = \begin{bmatrix} e^x & -3e^{2x} \\ 3e^x & e^{2x} \end{bmatrix}\frac{1}{10}\begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} e^x + 9e^{2x} & 3e^x - 3e^{2x} \\ 3e^x - 3e^{2x} & 9e^x + e^{2x} \end{bmatrix}.$$

(f) We can check that e^{Mx} equals the identity matrix if we set $x = 0$:

$$\frac{1}{10}\begin{bmatrix} e^x + 9e^{2x} & 3e^x - 3e^{2x} \\ 3e^x - 3e^{2x} & 9e^x + e^{2x} \end{bmatrix} \overset{x=0}{\rightsquigarrow} \frac{1}{10}\begin{bmatrix} 1 + 9 & 3 - 3 \\ 3 - 3 & 9 + 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This check does not require much effort and can even be done in our head while writing down e^{Mx}. There is really no excuse for not doing it!

Example 74. (homework) Let $M = \begin{bmatrix} -1 & 6 \\ -1 & 4 \end{bmatrix}$.

(a) Determine the general solution to $\boldsymbol{y}' = M\boldsymbol{y}$.

(b) Determine a fundamental matrix solution to $\boldsymbol{y}' = M\boldsymbol{y}$.

(c) Compute e^{Mx}.

(d) Solve the initial value problem $\boldsymbol{y}' = M\boldsymbol{y}$ with $\boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(e) Compute M^n.

(f) Solve $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$ with $\boldsymbol{a}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Solution.

(a) We determine the eigenvectors of M. The characteristic polynomial is:

$\det(M - \lambda I) = \det\left(\begin{bmatrix} -1-\lambda & 6 \\ -1 & 4-\lambda \end{bmatrix}\right) = (-1-\lambda)(4-\lambda) + 6 = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2)$

Hence, the eigenvalues are $\lambda = 1$ and $\lambda = 2$.

- $\lambda = 1$: Solving $\begin{bmatrix} -2 & 6 \\ -1 & 3 \end{bmatrix}\boldsymbol{v} = \boldsymbol{0}$, we find that $\boldsymbol{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 1$.
- $\lambda = 2$: Solving $\begin{bmatrix} -3 & 6 \\ -1 & 2 \end{bmatrix}\boldsymbol{v} = \boldsymbol{0}$, we find that $\boldsymbol{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 2$.

Hence, the general solution is $C_1\begin{bmatrix} 3 \\ 1 \end{bmatrix}e^x + C_2\begin{bmatrix} 2 \\ 1 \end{bmatrix}e^{2x}$.

(b) The corresponding fundamental matrix solution is $\Phi = \begin{bmatrix} 3e^x & 2e^{2x} \\ e^x & e^{2x} \end{bmatrix}$.

(c) Note that $\Phi(0) = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, so that $\Phi(0)^{-1} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$. It follows that

$$e^{Mx} = \Phi(x)\Phi(0)^{-1} = \begin{bmatrix} 3e^x & 2e^{2x} \\ e^x & e^{2x} \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 3e^x - 2e^{2x} & -6e^x + 6e^{2x} \\ e^x - e^{2x} & -2e^x + 3e^{2x} \end{bmatrix}.$$

(d) The solution to the IVP is $\boldsymbol{y}(x) = e^{Mx}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3e^x - 2e^{2x} & -6e^x + 6e^{2x} \\ e^x - e^{2x} & -2e^x + 3e^{2x} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3e^x + 4e^{2x} \\ -e^x + 2e^{2x} \end{bmatrix}$.

Note. If we hadn't already computed e^{Mx}, we would use the general solution and solve for the appropriate values of C_1 and C_2. Do it that way as well!

(e) From the first part, it follows that $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$ has general solution $C_1\begin{bmatrix} 3 \\ 1 \end{bmatrix} + C_2\begin{bmatrix} 2 \\ 1 \end{bmatrix}2^n$.

(Note that $1^n = 1$.)

The corresponding fundamental matrix solution is $\Phi_n = \begin{bmatrix} 3 & 2 \cdot 2^n \\ 1 & 2^n \end{bmatrix}$.

As above, $\Phi_0 = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, so that $\Phi(0)^{-1} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$ and

$$M^n = \Phi_n\Phi_0^{-1} = \begin{bmatrix} 3 & 2 \cdot 2^n \\ 1 & 2^n \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 3 - 2 \cdot 2^n & -6 + 6 \cdot 2^n \\ 1 - 2^n & -2 + 3 \cdot 2^n \end{bmatrix}.$$

Important. Compare with our computation for e^{Mx}. Can you see how this was basically the same computation? Write down M^n directly from e^{Mx}.

(f) The (unique) solution is $\boldsymbol{a}_n = M^n\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 - 2 \cdot 2^n & -6 + 6 \cdot 2^n \\ 1 - 2^n & -2 + 3 \cdot 2^n \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 + 4 \cdot 2^n \\ -1 + 2 \cdot 2^n \end{bmatrix}$.

Important. Again, compare with the earlier IVP! Without work, we can write down one from the other.

Example 75. (review) Let $M = \begin{bmatrix} -1 & 6 \\ -1 & 4 \end{bmatrix}$.

(a) Compute e^{Mx}.

(b) Solve the initial value problem $\boldsymbol{y}' = M\boldsymbol{y}$ with $\boldsymbol{y}(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

Solution.

(a) We determine the eigenvectors of M. The characteristic polynomial is:
$\det(M - \lambda I) = \det\left(\begin{bmatrix} -1-\lambda & 6 \\ -1 & 4-\lambda \end{bmatrix}\right) = (-1-\lambda)(4-\lambda) + 6 = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2)$
Hence, the eigenvalues are $\lambda = 1$ and $\lambda = 2$.

- $\lambda = 1$: Solving $\begin{bmatrix} -2 & 6 \\ -1 & 3 \end{bmatrix}\boldsymbol{v} = \boldsymbol{0}$, we find that $\boldsymbol{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 1$.
- $\lambda = 2$: Solving $\begin{bmatrix} -3 & 6 \\ -1 & 2 \end{bmatrix}\boldsymbol{v} = \boldsymbol{0}$, we find that $\boldsymbol{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 2$.

Hence, a fundamental matrix solution is $\Phi = \begin{bmatrix} 3e^x & 2e^{2x} \\ e^x & e^{2x} \end{bmatrix}$.

Note that $\Phi(0) = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, so that $\Phi(0)^{-1} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$. It follows that

$$e^{Mx} = \Phi(x)\Phi(0)^{-1} = \begin{bmatrix} 3e^x & 2e^{2x} \\ e^x & e^{2x} \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 3e^x - 2e^{2x} & -6e^x + 6e^{2x} \\ e^x - e^{2x} & -2e^x + 3e^{2x} \end{bmatrix}.$$

(b) The solution to the IVP is $\boldsymbol{y}(x) = e^{Mx}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 3e^x - 2e^{2x} \\ e^x - e^{2x} \end{bmatrix}$ (twice the first column of e^{Mx}).

Another perspective on the matrix exponential

(exponential function) e^x is the unique solution to $y' = y$, $y(0) = 1$.
From here, it follows that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$.

The latter is the Taylor series for e^x at $x = 0$ that we have seen in Calculus II.
Important note. We can actually construct this infinite sum directly from $y' = y$ and $y(0) = 1$.
Indeed, observe how each term, when differentiated, produces the term before it. For instance, $\frac{\mathrm{d}}{\mathrm{d}x}\frac{x^3}{3!} = \frac{x^2}{2!}$.

Review. We defined the **matrix exponential** e^{Mx} as the unique matrix solution to the IVP

$$\boldsymbol{y}' = M\boldsymbol{y}, \quad \boldsymbol{y}(0) = I.$$

Below, we observe that we can also make sense of the matrix exponential e^{Mx} as a power series.

Theorem 76. Let M be $n\times n$. Then the **matrix exponential** satisfies

$$e^M = I + M + \frac{1}{2!}M^2 + \frac{1}{3!}M^3 + \ldots$$

Proof. Define $\Phi(x) = I + Mx + \frac{1}{2!}M^2x^2 + \frac{1}{3!}M^3x^3 + \ldots$

$$\begin{aligned}\Phi'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left[I + Mx + \frac{1}{2!}M^2x^2 + \frac{1}{3!}M^3x^3 + \ldots\right]\\ &= 0 + M + M^2x + \frac{1}{2!}M^3x^2 + \ldots = M\Phi(x).\end{aligned}$$

Clearly, $\Phi(0) = I$. Therefore, $\Phi(x)$ is the fundamental matrix solution to $\boldsymbol{y}' = M\boldsymbol{y}$, $\boldsymbol{y}(0) = I$. But that's precisely how we defined e^{Mx} earlier. It follows that $\Phi(x) = e^{Mx}$. Now set $x = 1$. □

Example 77. If $A = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$, then $A^{100} = \begin{bmatrix} 2^{100} & 0 \\ 0 & 5^{100} \end{bmatrix}$.

Example 78. If $A = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$, then $e^A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} 2^2 & 0 \\ 0 & 5^2 \end{bmatrix} + \cdots = \begin{bmatrix} e^2 & 0 \\ 0 & e^5 \end{bmatrix}$.

Clearly, this works to obtain e^D for every diagonal matrix D.

In particular, for $Ax = \begin{bmatrix} 2x & 0 \\ 0 & 5x \end{bmatrix}$, $e^{Ax} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2x & 0 \\ 0 & 5x \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} (2x)^2 & 0 \\ 0 & (5x)^2 \end{bmatrix} + \cdots = \begin{bmatrix} e^{2x} & 0 \\ 0 & e^{5x} \end{bmatrix}$.

Extra: The case of repeated eigenvalues with too few eigenvectors

Review. To construct a fundamental matrix solution $\Phi(x)$ to $\boldsymbol{y}' = M\boldsymbol{y}$, we compute eigenvectors: Given a λ-eigenvector $\boldsymbol{v}$, we have the corresponding solution $\boldsymbol{y}(x) = \boldsymbol{v}e^{\lambda x}$.

If there are enough eigenvectors, we can collect these as columns to obtain $\Phi(x)$.

The next example illustrates how to proceed if there are not enough eigenvectors.

In that case, instead of looking only for solutions of the type $\boldsymbol{y}(x) = \boldsymbol{v}e^{\lambda x}$, we also need to look for solutions of the type $\boldsymbol{y}(x) = (\boldsymbol{v}x + \boldsymbol{w})e^{\lambda x}$. This can only happen if an eigenvalue is a repeated root of the characteristic polynomial.

Example 79. Let $M = \begin{bmatrix} 8 & 4 \\ -1 & 4 \end{bmatrix}$.

(a) Determine the general solution to $\boldsymbol{y}' = M\boldsymbol{y}$.

(b) Determine a fundamental matrix solution to $\boldsymbol{y}' = M\boldsymbol{y}$.

(c) Compute e^{Mx}.

(d) Solve the initial value problem $\boldsymbol{y}' = M\boldsymbol{y}$ with $\boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Armin Straub
straub@southalabama.edu

Solution.

(a) We determine the eigenvectors of M. The characteristic polynomial is:

$\det(M-\lambda I)=\det\left(\begin{bmatrix} 8-\lambda & 4 \\ -1 & 4-\lambda \end{bmatrix}\right)=(8-\lambda)(4-\lambda)+4=\lambda^2-12\lambda+36=(\lambda-6)(\lambda-6)$

Hence, the eigenvalues are $\lambda=6,6$ (meaning that 6 has multiplicity 2).

- To find eigenvectors $\boldsymbol{v}$ for $\lambda=6$, we need to solve $\begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}\boldsymbol{v}=\boldsymbol{0}$.

  Hence, $\boldsymbol{v}=\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda=6$. There is no independent second eigenvector.

- We therefore search for a solution of the form $\boldsymbol{y}(x)=(\boldsymbol{v}x+\boldsymbol{w})e^{\lambda x}$ with $\lambda=6$.

  $\boldsymbol{y}'(x)=(\lambda\boldsymbol{v}x+\lambda\boldsymbol{w}+\boldsymbol{v})e^{\lambda x}\stackrel{!}{=}M\boldsymbol{y}=(M\boldsymbol{v}x+M\boldsymbol{w})e^{\lambda x}$

  Equating coefficients of x, we need $\lambda\boldsymbol{v}=M\boldsymbol{v}$ and $\lambda\boldsymbol{w}+\boldsymbol{v}=M\boldsymbol{w}$.

  Hence, $\boldsymbol{v}$ must be an eigenvector (which we already computed); we choose $\boldsymbol{v}=\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

  [Note that any multiple of $\boldsymbol{y}(x)$ will be another solution, so it doesn't matter which multiple of $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ we choose.]

  $\lambda\boldsymbol{w}+\boldsymbol{v}=M\boldsymbol{w}$ or $(M-\lambda)\boldsymbol{w}=\boldsymbol{v}$ then becomes $\begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}\boldsymbol{w}=\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

  One solution is $\boldsymbol{w}=\begin{bmatrix} -1 \\ 0 \end{bmatrix}$. [We only need one.]

Hence, the general solution is $C_1\begin{bmatrix} -2 \\ 1 \end{bmatrix}e^{6x}+C_2\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}x+\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right)e^{6x}$.

(b) The corresponding fundamental matrix solution is $\Phi=\begin{bmatrix} -2e^{6x} & -(2x+1)e^{6x} \\ e^{6x} & x\,e^{6x} \end{bmatrix}$.

(c) Note that $\Phi(0)=\begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix}$, so that $\Phi(0)^{-1}=\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$. It follows that

$$e^{Mx}=\Phi(x)\Phi(0)^{-1}=\begin{bmatrix} -2e^{6x} & -(2x+1)e^{6x} \\ e^{6x} & x\,e^{6x} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}=\begin{bmatrix} (2x+1)e^{6x} & 4x\,e^{6x} \\ -x\,e^{6x} & -(2x-1)\,e^{6x} \end{bmatrix}.$$

(d) The solution to the IVP is $\boldsymbol{y}(x)=e^{Mx}\begin{bmatrix} 1 \\ 0 \end{bmatrix}=\begin{bmatrix} (2x+1)e^{6x} & 4x\,e^{6x} \\ -x\,e^{6x} & -(2x-1)\,e^{6x} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}=\begin{bmatrix} (2x+1)e^{6x} \\ -x\,e^{6x} \end{bmatrix}$.

Armin Straub
straub@southalabama.edu

Phase portraits and phase plane analysis

Our goal is to visualize the solutions to systems of equations. This works particularly well in the case of systems of two differential equations.

A system of two differential equations that can be written as

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= f(x,y)\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= g(x,y)\end{aligned}$$

is called **autonomous** because it doesn't depend on the independent variable t.

Comment. Can you show that if $x(t)$ and $y(t)$ are a pair of solutions, then so is the pair $x(t+t_0)$ and $y(t+t_0)$?

We can visualize solutions to such a system by plotting the points $(x(t), y(t))$ for increasing values of t so that we get a curve (and we can attach an arrow to indicate the direction we're flowing along that curve). Each such curve is called the **trajectory** of a solution.

Even better, we can do such a **phase portrait** without solving to get a formula for $(x(t), y(t))$! That's because we can combine the two equations to get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{g(x,y)}{f(x,y)},$$

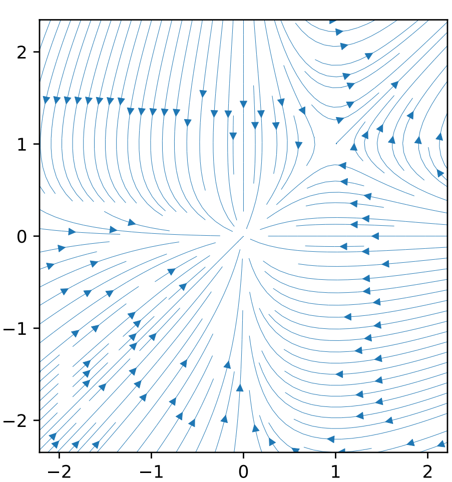
which allows us to make a slope field! If a trajectory passes through a point (x,y), then we know that the slope at that point must be $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{g(x,y)}{f(x,y)}$.

This allows us to sketch trajectories. However, it does not tell us everything about the corresponding solution $(x(t), y(t))$ because we don't know at which times t the solution passes through the points on the curve.

However, we can visualize the speed with which a solution passes through the trajectory by attaching to a point (x,y) not only the slope $\frac{g(x,y)}{f(x,y)}$ but the vector $\begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$. That vector has the same direction as the slope but it also tells us in which direction we are moving and how fast (by its magnitude).

Example 80. Sketch some trajectories for the system $\frac{\mathrm{d}x}{\mathrm{d}t} = x\cdot(y-1)$, $\frac{\mathrm{d}y}{\mathrm{d}t} = y\cdot(x-1)$.

Solution.



Armin Straub
straub@southalabama.edu

Comment. In this example, we can solve the slope-field equation $\frac{dy}{dx} = \frac{y(x-1)}{x(y-1)}$ using separation of variables.

Do it! We end up with the implicit solutions $y - \ln|y| = x - \ln|x| + C$.

If we plot these curves for various values of C, we get trajectories in the plot above. However, note that none of this solving is needed for plotting by itself.

Sage. We can make Sage create such phase portraits for us!

```
>>> x,y = var('x y')
```

```
>>> plot = streamline_plot((x*(y-1),y*(x-1)), (x,-3,3), (y,-3,3))
```

Equilibrium solutions

(x_0, y_0) is an **equilibrium point** of the system $\frac{dx}{dt} = f(x,y)$, $\frac{dy}{dt} = g(x,y)$ if

$$f(x_0, y_0) = 0 \quad \text{and} \quad g(x_0, y_0) = 0.$$

In that case, we have the **equilibrium solution** $x(t) = x_0$, $y(t) = y_0$.

Comment. Equilibrium points are also called **critical points** (or stationary points or rest points).

In a phase portrait, the equilibrium solutions are just a single point.

Recall that every other solution $(x(t), y(t))$ corresponds to a curve (parametrized by t), called the **trajectory** of the solution (and we can adorn it with an arrow that indicates the direction of the "flow" of the solution).

We can learn a lot from how solutions behave near equilibrium points.

An equilibrium point is called:

- **stable** if all nearby solutions remain close to the equilibrium point;
- **asymptotically stable** if all nearby solutions remain close and "flow into" the equilibrium;
- **unstable** if it is not stable (some nearby solutions "flow away" from the equilibrium).

Comment. Note that asymptotically stable is a stronger condition than stable. A typical example of a stable, but not asymptotically stable, equilibrium point is one where nearby solutions loop around the equilibrium point without coming closer to it.

Advanced comment. For asymptotically stable, we kept the condition that nearby solutions remain close because there are "weird" instances where trajectories come arbitrarily close to the equilibrium, then "flow away" but eventually "flow into" (this would constitute an unstable equilibrium point).

Example 81. (cont'd) Consider again the system $\frac{dx}{dt} = x \cdot (y-1)$, $\frac{dy}{dt} = y \cdot (x-1)$.

(a) Determine the equilibrium points.

(b) Using the phase portrait from Example 80, classify the stability of each equilibrium point.

Solution.

(a) We solve $x(y-1) = 0$ (that is, $x = 0$ or $y = 1$) and $y(x-1) = 0$ (that is, $x = 1$ or $y = 0$).
We conclude that the equilibrium points are $(0,0)$ and $(1,1)$.

(b) $(0,0)$ is asymptotically stable (because all nearby solutions "flow into" $(0,0)$).
$(1,1)$ is unstable (because some nearby solutions "flow away" from $(1,1)$).
Comment. We will soon learn how to determine stability without the need for a plot.
Comment. If you look carefully at the phase portrait near $(1,1)$, you can see that certain solutions get attracted at first to $(1,1)$ and then "flow away" at the last moment. This suggests that there is a single trajectory which actually "flows into" $(1,1)$. This constellation is typical and is called a **saddle point**.

Example 82. Consider the system $\frac{dx}{dt} = y - 5x$, $\frac{dy}{dt} = 4x - 2y$.

(a) Determine the general solution.

(b) Make a phase portrait. Can you connect it with the general solution?

(c) Determine all equilibrium points and their stability.

Solution.

(a) Note that we can write this is in matrix form as $\begin{bmatrix} x \\ y \end{bmatrix}' = M \begin{bmatrix} x \\ y \end{bmatrix}$ with $M = \begin{bmatrix} -5 & 1 \\ 4 & -2 \end{bmatrix}$.

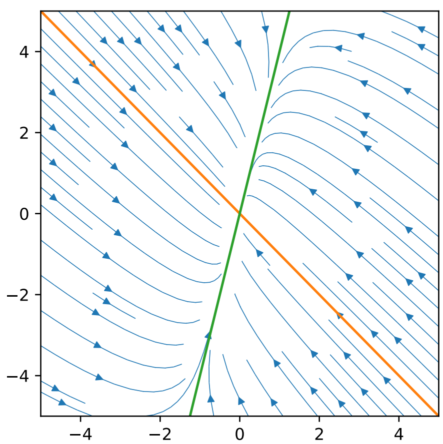
M has -1-eigenvector $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ as well as -6-eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

The general solution is $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-6t}$.

(b) We can have Sage make such a plot for us:

```
>>> x,y = var('x y')
    plot = streamline_plot((-5*x+y,4*x-2*y), (x,-4,4), (y,-4,4))
```

Question. In our plot, we also highlighted two lines through the origin. Can you explain their significance?



Explanation. The two lines correspond to the special solutions $C_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{-t}$ and $C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-6t}$. In each case, the trajectories consist of points that are multiples of the vectors $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, respectively.

Note that each such solution starts at a point on one of the lines and then "flows" into the origin. (Because e^{-t} and e^{-6t} approach zero for large t.)

Question. Consider a point like $(4, 4)$. Can you explain why the trajectory through that point doesn't go somewhat straight to $(0, 0)$ but rather flows nearly parallel the orange line towards the green line?

Explanation. A solution through $(4, 4)$ is of the form $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-6t}$ (like any other solution). Note that, if we increase t, then e^{-6t} becomes small much faster than e^{-t}.

As a consequence, we quickly get $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \approx C_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{-t}$, where the right-hand side is on the green line.

(c) The only equilibrium point is $(0, 0)$ and it is asymptotically stable.

We can see this from the phase portrait but we can also determine it from the DE and our solution: first, solving $y - 5x = 0$ and $4x - 2y = 0$ we only get the unique solution $x = 0, y = 0$, which means that only $(0, 0)$ is an equilibrium point. On the other hand, the general solution shows that every solution approaches $(0, 0)$ as $t \to \infty$ because both e^{-t} and e^{-6t} approach 0.

In general. This is typical: if both eigenvalues are negative, then the equilibrium is asymptotically stable. If at least one eigenvalue is positive, then the equilibrium is unstable.

Example 83. Consider the system $\frac{dx}{dt} = 5x - y$, $\frac{dy}{dt} = 2y - 4x$.

(a) Determine the general solution.

(b) Make a phase portrait.

(c) Determine all equilibrium points and their stability.

Solution.

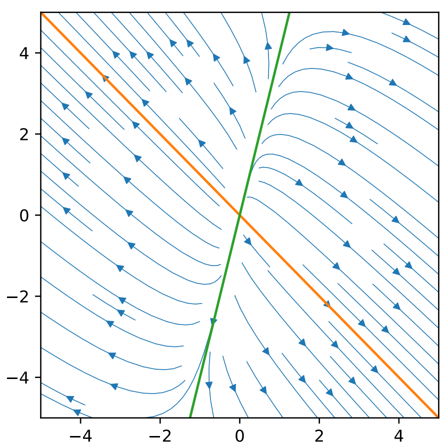
(a) Note that we can write this is in matrix form as $\begin{bmatrix} x \\ y \end{bmatrix}' = M \begin{bmatrix} x \\ y \end{bmatrix}$ with $M = -\begin{bmatrix} -5 & 1 \\ 4 & -2 \end{bmatrix}$, where the matrix is exactly -1 times what it was in Example 82.

Consequently, M has 1-eigenvector $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ as well as 6-eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ (can you explain why the eigenvectors are the same and the eigenvalues changed sign?).

Thus, the general solution is $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^t + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{6t}$.

(b) We can have Sage make the plot for us:

```
>>> x,y = var('x y')
    plot = streamline_plot((5*x-y,-4*x+2*y), (x,-4,4), (y,-4,4))
```



Note that the phase portrait is identical to the one in Example 82, except that the arrows are reversed.

(c) The only equilibrium point is $(0,0)$ and it is unstable.

We can see this from the phase portrait but we can also see it readily from our general solution $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^t + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{6t}$ because e^t and e^{6t} go to ∞ as $t \to \infty$.

In general. If at least one eigenvalue is positive, then the equilibrium is unstable.

Example 84. Suppose the system $\frac{dx}{dt} = f(x,y)$, $\frac{dy}{dt} = g(x,y)$ has general solution $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{6t}$. Determine all equilibrium points and their stability.

Solution. Clearly, the only constant solution is the zero solution $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Equivalently, the only equilibrium point is $(0,0)$.

Since $e^{6t} \to \infty$ as $t \to \infty$, we conclude that the equilibrium is unstable. (Note that the solution $C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{6t}$ starts arbitrarily near to $(0,0)$ but always "flows away").

Armin Straub
straub@southalabama.edu

Excursion: Euler's identity

Theorem 85. (Euler's identity) $e^{ix} = \cos(x) + i\sin(x)$

Proof. Observe that both sides are the (unique) solution to the IVP $y' = iy$, $y(0) = 1$.
[Check that by computing the derivatives and verifying the initial condition! As we did in class.] $\square$

On lots of T-shirts. In particular, with $x = \pi$, we get $\boxed{e^{\pi i} = -1}$ or $e^{i\pi} + 1 = 0$ (which connects the five fundamental constants).

Example 86. Where do trig identities like $\sin(2x) = 2\cos(x)\sin(x)$ or $\sin^2(x) = \frac{1 - \cos(2x)}{2}$ (and infinitely many others you have never heard of!) come from?

Short answer: they all come from the simple exponential law $e^{x+y} = e^x e^y$.

Let us illustrate this in the simple case $(e^x)^2 = e^{2x}$. Observe that

$$\begin{aligned} e^{2ix} &= \cos(2x) + i\sin(2x) \\ e^{ix}e^{ix} &= [\cos(x) + i\sin(x)]^2 = \cos^2(x) - \sin^2(x) + 2i\cos(x)\sin(x). \end{aligned}$$

Comparing imaginary parts (the "stuff with an i"), we conclude that $\sin(2x) = 2\cos(x)\sin(x)$.
Likewise, comparing real parts, we read off $\cos(2x) = \cos^2(x) - \sin^2(x)$.

(Use $\cos^2(x) + \sin^2(x) = 1$ to derive $\sin^2(x) = \frac{1 - \cos(2x)}{2}$ from the last equation.)

Challenge. Can you find a triple-angle trig identity for $\cos(3x)$ and $\sin(3x)$ using $(e^x)^3 = e^{3x}$?
Or, use $e^{i(x+y)} = e^{ix}e^{iy}$ to derive $\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ and $\sin(x + y) = \ldots$ (that's what we actually did in class).

Realize that the complex number $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ corresponds to the point $(\cos(\theta), \sin(\theta))$. These are precisely the points on the unit circle!

Recall that a point (x, y) can be represented using **polar coordinates** (r, θ), where r is the distance to the origin and θ is the angle with the x-axis.

Then, $x = r\cos\theta$ and $y = r\sin\theta$.

Every complex number z can be written in **polar form** as $z = re^{i\theta}$, with $r = |z|$.

Why? By comparing with the usual polar coordinates ($x = r\cos\theta$ and $y = r\sin\theta$), we can write

$$z = x + iy = r\cos\theta + ir\sin\theta = re^{i\theta}.$$

In the final step, we used Euler's identity.

Armin Straub
straub@southalabama.edu

Stability of autonomous linear differential equations

Example 87. (cont'd) Analyze the system $\frac{\mathrm{d}x}{\mathrm{d}t} = y - 5x + 3$, $\frac{\mathrm{d}y}{\mathrm{d}t} = 4x - 2y$.

In particular, determine the general solution as well as all equilibrium points and their stability.

Solution. Note that in Example 82, we looked at the corresponding homogeneous system and found that its general solution is $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-6t}$.

Note that we can write the present system in matrix form as $\begin{bmatrix} x \\ y \end{bmatrix}' = M \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ with $M = \begin{bmatrix} -5 & 1 \\ 4 & -2 \end{bmatrix}$.

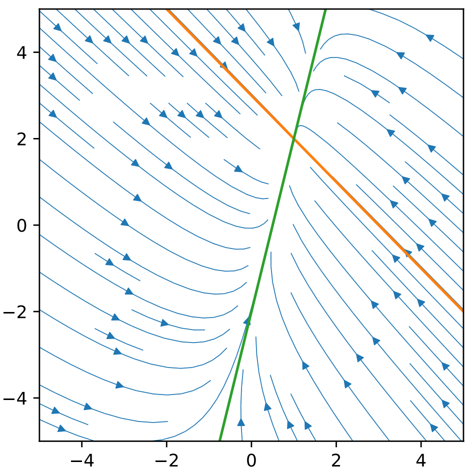
To find the equilibrium point, we solve $M \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} = 0$ to find $\begin{bmatrix} x \\ y \end{bmatrix} = -M^{-1} \begin{bmatrix} 3 \\ 0 \end{bmatrix} = -\frac{1}{6} \begin{bmatrix} -2 & -1 \\ -4 & -5 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

The fact that $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an equilibrium point means that $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is a particular solution!

(Make sure that you see that it has exactly the form we expect from the method of undetermined coefficients!)

Thus, the general solution must be $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-6t}$.

As a result, the phase portrait is going to look just as in Example 82 but shifted by $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$:



Because both eigenvalues (-1 and -6) are negative, $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an asymptotically stable equilibrium point. More precisely, it is what is called a **nodal source**.

As we have started to observe, the eigenvalues determine the stability of the equilibrium point in the case of an autonomous linear 2-dimensional systems. The following table gives an overview.

Important. Note that such a system must be of the form $\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} x \\ y \end{bmatrix} = M \begin{bmatrix} x \\ y \end{bmatrix} + \boldsymbol{c}$, where $\boldsymbol{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ is a constant vector. Because the system is autonomous, the matrix M and the inhomogeneous part $\boldsymbol{c}$ cannot depend on t.

(stability of autonomous linear 2-dimensional systems)

eigenvalues	**behaviour**	**stability**
real and both positive	nodal source	unstable
real and both negative	nodal sink	asymptotically stable
real and opposite signs	saddle	unstable
complex with positive real part	spiral source	unstable
complex with negative real part	spiral sink	asymptotically stable
purely imaginary	center point	stable but not asymptotically stable

Example 88. (spiral source, spiral sink, center point)

(a) Analyze the system $\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

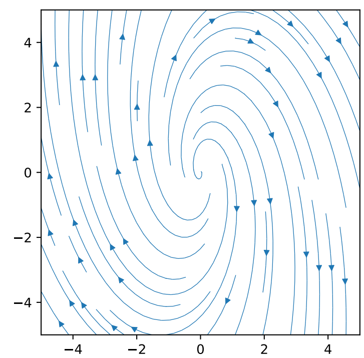
(b) Analyze the system $\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \end{bmatrix} = -\begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

(c) Analyze the system $\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

Solution.

(a) The general solution, in real form, is:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1\begin{bmatrix} \cos(2t) \\ -2\sin(2t) \end{bmatrix} e^t + C_2\begin{bmatrix} \sin(2t) \\ 2\cos(2t) \end{bmatrix} e^t$$



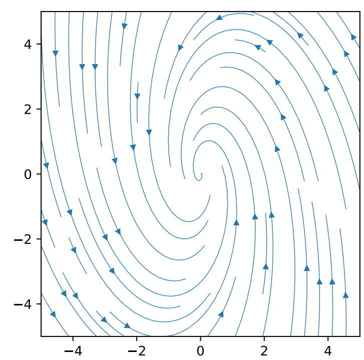
In this case, the origin is a spiral source which is an unstable equilibrium (note that it follows from $e^t \to \infty$ as $t \to \infty$ that all solutions "flow away" from the origin because they have increasing amplitude).

Review. $\begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$ parametrizes the unit circle.

Similarly, $\begin{bmatrix} \cos(t) \\ 2\sin(t) \end{bmatrix}$ parametrizes an ellipse.

(b) The general solution, in real form, is:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1\begin{bmatrix} \cos(2t) \\ -2\sin(2t) \end{bmatrix} e^{-t} + C_2\begin{bmatrix} \sin(2t) \\ 2\cos(2t) \end{bmatrix} e^{-t}$$

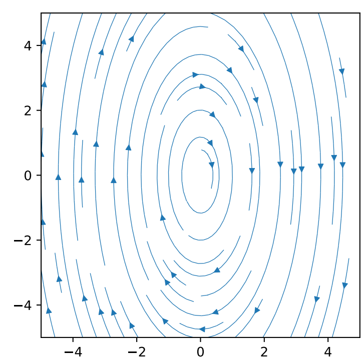


In this case, the origin is a spiral sink which is an asymptotically stable equilibrium (note that it follows from $e^{-t} \to 0$ as $t \to \infty$ that all solutions "flow into" the origin because their amplitude goes to zero).

Comment. Note that $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ solves the first system if and only if $\begin{bmatrix} x(-t) \\ y(-t) \end{bmatrix}$ is a solution to the second. Consequently, the phase portraits look alike but all arrows are reversed.

(c) The general solution, in real form, is:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1\begin{bmatrix} \cos(2t) \\ -2\sin(2t) \end{bmatrix} + C_2\begin{bmatrix} \sin(2t) \\ 2\cos(2t) \end{bmatrix}$$



In this case, the origin is a center point which is a stable equilibrium (note that the solutions are periodic with period π and therefore loop around the origin; with each trajectory a perfect ellipse).

Review: Linearizations of nonlinear functions

Recall from Calculus I that a function $f(x)$ around a point x_0 has the linearization

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0).$$

Here, the right-hand side is the linearization and we also know it as the tangent line to $f(x)$ at x_0.

Recall from Calculus III that a function $f(x, y)$ around a point (x_0, y_0) has the linearization

$$f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Again, the right-hand side is the linearization. This time, it describes the tangent plane to $f(x, y)$ at (x_0, y_0). Recall that $f_x = \frac{\partial}{\partial x} f(x, y)$ and $f_y = \frac{\partial}{\partial y} f(x, y)$ are the partial derivatives of f.

Example 89. Determine the linearization of the function $3 + 2xy^2$ at $(2, 1)$.

Solution. If $f(x, y) = 3 + 2xy^2$, then $f_x = 2y^2$ and $f_y = 4xy$. In particular, $f_x(2, 1) = 2$ and $f_y(2, 1) = 8$. Accordingly, the linearization is $f(2, 1) + f_x(2, 1)(x - 2) + f_y(2, 1)(y - 1) = 7 + 2(x - 2) + 8(y - 1)$.

It follows that a vector function $\boldsymbol{f}(x, y) = \begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix}$ around a point (x_0, y_0) has the linearization

$$\begin{aligned}
\begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix} &\approx \begin{bmatrix} f(x_0, y_0) \\ g(x_0, y_0) \end{bmatrix} + \begin{bmatrix} f_x(x_0, y_0) \\ g_x(x_0, y_0) \end{bmatrix}(x - x_0) + \begin{bmatrix} f_y(x_0, y_0) \\ g_y(x_0, y_0) \end{bmatrix}(y - y_0) \\
&= \begin{bmatrix} f(x_0, y_0) \\ g(x_0, y_0) \end{bmatrix} + \underbrace{\begin{bmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{bmatrix}}_{= J(x_0, y_0)} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}.
\end{aligned}$$

The matrix $J(x, y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}$ is called the **Jacobian matrix** of $\boldsymbol{f}(x, y)$.

Example 90. Determine the linearization of the vector function $\begin{bmatrix} 3 + 2xy^2 \\ x(y^3 - 2x) \end{bmatrix}$ at $(2, 1)$.

Solution. If $\begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix} = \begin{bmatrix} 3 + 2xy^2 \\ x(y^3 - 2x) \end{bmatrix}$, then the Jacobian matrix is

$$J(x, y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} 2y^2 & 4xy \\ y^3 - 4x & 3xy^2 \end{bmatrix}.$$

In particular, $J(2, 1) = \begin{bmatrix} 2 & 8 \\ -7 & 6 \end{bmatrix}$. The linearization is $\begin{bmatrix} f(2, 1) \\ g(2, 1) \end{bmatrix} + J(2, 1)\begin{bmatrix} x - 2 \\ y - 1 \end{bmatrix} = \begin{bmatrix} 7 \\ -6 \end{bmatrix} + \begin{bmatrix} 2 & 8 \\ -7 & 6 \end{bmatrix}\begin{bmatrix} x - 2 \\ y - 1 \end{bmatrix}$.

Important comment. If we multiply out the matrix-vector product, then we get $\begin{bmatrix} 7 + 2(x - 2) + 8(y - 1) \\ -6 - 7(x - 2) + 6(y - 1) \end{bmatrix}$.

In the first component we get exactly what we got for the linearization of $f(x, y)$ in the previous example. Likewise, the second component is the linearization of $g(x, y)$ by itself.

Stability of nonlinear autonomous systems

We now observe that we can (typically) determine the stability of an equilibrium point of a nonlinear system by simply linearizing at that point.

(stability of autonomous nonlinear 2-dimensional systems)

Suppose that (x_0, y_0) is an equilibrium point of the system $\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$.

If the Jacobian matrix $J(x_0, y_0)$ is invertible, then its eigenvalues determine the stability and behaviour of the equlibrium point as for a linear system except in the following cases:

- If the eigenvalues are pure imaginary, we cannot predict stability (the equilibrium point could be either a center or a spiral source/sink; whereas the equilibrium point of the linearization is a center).
- If the eigenvalues are real and equal, then the equilibrium point could be either nodal or spiral (whereas the linearization has a nodal equilibrium point). The stability, however, is the same.

Comment. We need the Jacobian matrix $J(x_0, y_0)$ to be invertible, so that the linearized system has a unique equilibrium point.

Plot, for instance, the phase portrait of $\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (x-2y)x \\ (x-2)y \end{bmatrix}$.

Purely imaginary eigenvalues? The issue with pure imaginary eigenvalues here comes from the fact that the linearization is only an approximation, with the true (nonlinear) behaviour slightly deviating. Slightly perturbing purely imaginary roots can also lead to (small but) positive real part (unstable; spiral source) or negative real part (asymptotically stable; spiral sink).

Real repeated eigenvalue? The issue with a real repeated eigenvalue is similar. Slightly perturbing such a root can lead to real eigenvalues (nodal) or a pair of complex conjugate eigenvalues (spiral). However, the real part of these perturbations still has the same sign so that we can still predict the stability itself.

Example 91. (cont'd) Consider again the system $\frac{\mathrm{d}x}{\mathrm{d}t} = x \cdot (y-1)$, $\frac{\mathrm{d}y}{\mathrm{d}t} = y \cdot (x-1)$. Without consulting a plot, determine the equilibrium points and classify their stability.

Solution. See Example 80 for the phase portrait. However, we will not use it in the following.

To find the equilibrium points, we solve $x(y-1) = 0$ (that is, $x = 0$ or $y = 1$) and $y(x-1) = 0$ (that is, $x = 1$ or $y = 0$). We conclude that the equilibrium points are $(0,0)$ and $(1,1)$.

Our system is $\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$ with $\begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} = \begin{bmatrix} x \cdot (y-1) \\ y \cdot (x-1) \end{bmatrix}$.

The Jacobian matrix is $J(x,y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} y-1 & x \\ y & x-1 \end{bmatrix}$.

- At $(0,0)$, the Jacobian matrix is $J(0,0) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. We can read off that the eigenvalues are $-1, -1$. Since they are both negative, $(0,0)$ is a nodal sink. In particular, $(0,0)$ is asymptotically stable.
- At $(1,1)$, the Jacobian matrix is $J(1,1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

  The characteristic polynomial is $\det\left(\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}\right) = \lambda^2 - 1$, which has roots ± 1. These are the eigenvalues. Since one is positive and the other is negative, $(1,1)$ is a saddle. In particular, $(1,1)$ is unstable.

Example 92. Consider again the system $\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2y - x^2 \\ (x-3)(x-y) \end{bmatrix}$. Without consulting a plot, determine the equilibrium points and classify their stability.

Solution. To find the equilibrium points, we solve $2y - x^2 = 0$ and $(x-3)(x-y)$. It follows from the second equation that $x = 3$ or $x = y$:

- If $x = 3$, then the first equation implies $y = \frac{9}{2}$.
- If $x = y$, then the first equation becomes $2y - y^2 = 0$, which has solutions $y = 0$ and $y = 2$.

Hence, the equilibrium points are $(0,0)$, $(2,2)$ and $\left(3, \frac{9}{2}\right)$.

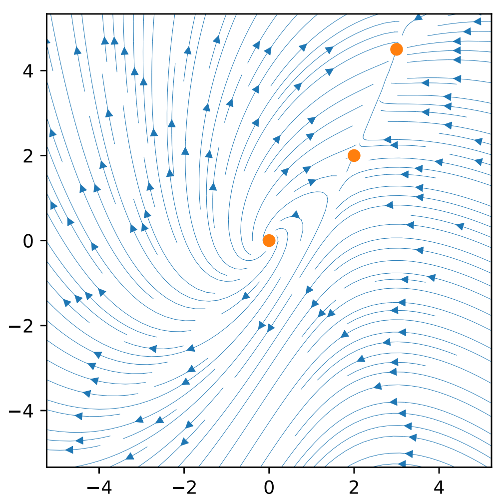
The Jacobian matrix of $\begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} 2y - x^2 \\ (x-3)(x-y) \end{bmatrix}$ is $J = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} -2x & 2 \\ 2x - y - 3 & -x + 3 \end{bmatrix}$.

- At $(0,0)$, the Jacobian matrix is $J = \begin{bmatrix} 0 & 2 \\ -3 & 3 \end{bmatrix}$. The eigenvalues are $\frac{1}{2}(3 \pm i\sqrt{15})$.

  Since these are complex with positive real part, $(0,0)$ is a spiral source and, in particular, unstable.
- At $(2,2)$, the Jacobian matrix is $J = \begin{bmatrix} -4 & 2 \\ -1 & 1 \end{bmatrix}$. The eigenvalues are $\frac{1}{2}(-3 \pm \sqrt{17}) \approx -3.562, 0.562$.

  Since these are real with opposite signs, $(2,2)$ is a saddle and, in particular, unstable.
- At $\left(3, \frac{9}{2}\right)$, the Jacobian matrix is $J = \begin{bmatrix} -6 & 2 \\ -\frac{3}{2} & 0 \end{bmatrix}$. The eigenvalues are $-3 \pm \sqrt{6} \approx -5.449, -0.551$.

  Since these are real and both negative, $\left(3, \frac{9}{2}\right)$ is a nodal sink and, in particular, asymptotically stable.



Comment. Can you confirm our analysis in the above plot? Note that it is becoming hard to see the details. One solution would be to make separate phase portraits focusing on the vicinity of each equilibrium plot. Do it!

Armin Straub
straub@southalabama.edu

Systems of linear DEs: the inhomogeneous case

Recall that any linear DE can be transformed into a first-order system. Hence, any linear DE (or any system of linear DEs) can be written as

$$\boldsymbol{y}' = A(t)\,\boldsymbol{y} + \boldsymbol{f}(t).$$

Note. In general, A depends on t. In other words, the DE is allowed to have nonconstant coefficients. On the other hand, this is an autonomous DE (those for which we can analyze phase portraits) only if $A(t)$ and $\boldsymbol{f}(t)$ actually don't depend on t.

Review. We showed in Theorem 18 that $y' = a(t)y + f(t)$ has the particular solution

$$y_p(t) = y_h(t)\int \frac{f(t)}{y_h(t)}\mathrm{d}t,$$

where $y_h(t) = e^{\int a(t)\mathrm{d}t}$ is any solution to the homogeneous equation $y' = a(t)y$.

The same arguments with the same result apply to systems of linear equations!

Theorem 93. (variation of constants) $\boldsymbol{y}' = A(t)\,\boldsymbol{y} + \boldsymbol{f}(t)$ has the particular solution

$$\boldsymbol{y}_p(t) = \Phi(t)\int \Phi(t)^{-1}\boldsymbol{f}(t)\mathrm{d}t,$$

where $\Phi(t)$ is any fundamental matrix solution to $\boldsymbol{y}' = A(t)\,\boldsymbol{y}$.

Proof. We can find this formula in the same manner as we did in Theorem 18:
Since the general solution of the homogeneous equation $\boldsymbol{y}' = A(t)\,\boldsymbol{y}$ is $\boldsymbol{y}_h = \Phi(t)\boldsymbol{c}$, we are going to vary the constant $\boldsymbol{c}$ and look for a particular solution of the form $\boldsymbol{y}_p = \Phi(t)\boldsymbol{c}(t)$. Plugging into the DE, we get:

$\boldsymbol{y}_p' = \Phi'\boldsymbol{c} + \Phi\boldsymbol{c}' = A\Phi\boldsymbol{c} + \Phi\boldsymbol{c}' \overset{!}{=} A\boldsymbol{y}_p + \boldsymbol{f} = A\Phi\boldsymbol{c} + \boldsymbol{f}$

For the first equality, we used the matrix version of the usual product rule (which holds since differentiation is defined entry-wise). For the second equality, we used $\Phi' = A\Phi$.
Hence, $\boldsymbol{y}_p = \Phi(t)\boldsymbol{c}(t)$ is a particular solution if and only if $\Phi\boldsymbol{c}' = \boldsymbol{f}$.
The latter condition means $\boldsymbol{c}' = \Phi^{-1}\boldsymbol{f}$ so that $\boldsymbol{c} = \int \Phi(t)^{-1}\boldsymbol{f}(t)\mathrm{d}t$, which gives the claimed formula for $\boldsymbol{y}_p$. $\square$

Example 94. Find a particular solution to $\boldsymbol{y}' = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}\boldsymbol{y} + \begin{bmatrix} 0 \\ -2e^{3t} \end{bmatrix}$.

Solution. First, we determine (do it!) a fundamental matrix solution for $\boldsymbol{y}' = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}\boldsymbol{y}$: $\Phi(x) = \begin{bmatrix} e^{-t} & 3e^{4t} \\ -e^{-t} & 2e^{4t} \end{bmatrix}$
Using $\det(\Phi(t)) = 5e^{3t}$, we find $\Phi(t)^{-1} = \frac{1}{5}\begin{bmatrix} 2e^{t} & -3e^{t} \\ e^{-4t} & e^{-4t} \end{bmatrix}$.

Hence, $\Phi(t)^{-1}\boldsymbol{f}(t) = \frac{2}{5}\begin{bmatrix} 3e^{4t} \\ -e^{-t} \end{bmatrix}$ and $\int \Phi(t)^{-1}\boldsymbol{f}(t)\mathrm{d}x = \frac{2}{5}\begin{bmatrix} 3/4e^{4t} \\ e^{-t} \end{bmatrix}$.

By variation of constants, $\boldsymbol{y}_p(t) = \Phi(t)\int \Phi(t)^{-1}\boldsymbol{f}(t)\mathrm{d}t = \begin{bmatrix} e^{-t} & 3e^{4t} \\ -e^{-t} & 2e^{4t} \end{bmatrix}\frac{2}{5}\begin{bmatrix} 3/4e^{4t} \\ e^{-t} \end{bmatrix} = \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix}e^{3t}$.

Comment. Note that the solution is of the form that we anticipate from the method of undetermined coefficients (which we only discussed in the case of a single DE but which works similarly for systems).

Armin Straub
straub@southalabama.edu

In the special case that $\Phi(t) = e^{At}$, some things become easier. For instance, $\Phi(t)^{-1} = e^{-At}$. In that case, we can explicitely write down solutions to IVPs:

- $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{c}$ has (unique) solution $\boldsymbol{y}(t) = e^{At}\boldsymbol{c}$.
- $\boldsymbol{y}' = A\boldsymbol{y} + \boldsymbol{f}(t)$, $\boldsymbol{y}(0) = \boldsymbol{c}$ has (unique) solution $\boldsymbol{y}(t) = e^{At}\boldsymbol{c} + e^{At}\int_0^t e^{-As}\boldsymbol{f}(s)\mathrm{d}s$.

Example 95. Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$.

(a) Determine e^{At}.

(b) Solve $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(c) Solve $\boldsymbol{y}' = A\boldsymbol{y} + \begin{bmatrix} 0 \\ 2e^t \end{bmatrix}$, $\boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Solution.

(a) By proceeding as in Example 74 (do it!), we find $e^{At} = \begin{bmatrix} 2e^{2t} - e^{3t} & -2e^{2t} + 2e^{3t} \\ e^{2t} - e^{3t} & -e^{2t} + 2e^{3t} \end{bmatrix}$.

(b) $\boldsymbol{y}(t) = e^{At}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2e^{2t} + 3e^{3t} \\ -e^{2t} + 3e^{3t} \end{bmatrix}$

(c) $\boldsymbol{y}(t) = e^{At}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + e^{At}\int_0^t e^{-As}\boldsymbol{f}(s)\mathrm{d}s$. We compute:

$\int_0^t e^{-As}\boldsymbol{f}(s)\mathrm{d}s = \int_0^t \begin{bmatrix} 2e^{-2s} - e^{-3s} & -2e^{-2s} + 2e^{-3s} \\ e^{-2s} - e^{-3s} & -e^{-2s} + 2e^{-3s} \end{bmatrix}\begin{bmatrix} 0 \\ 2e^s \end{bmatrix}\mathrm{d}s = \int_0^t \begin{bmatrix} -4e^{-s} + 4e^{-2s} \\ -2e^{-s} + 4e^{-2s} \end{bmatrix}\mathrm{d}s = \begin{bmatrix} 4e^{-t} - 2e^{-2t} - 2 \\ 2e^{-t} - 2e^{-2t} \end{bmatrix}$

Hence, $e^{At}\int_0^t e^{-As}\boldsymbol{f}(s)\mathrm{d}s = \begin{bmatrix} 2e^{2t} - e^{3t} & -2e^{2t} + 2e^{3t} \\ e^{2t} - e^{3t} & -e^{2t} + 2e^{3t} \end{bmatrix}\begin{bmatrix} 4e^{-t} - 2e^{-2t} - 2 \\ 2e^{-t} - 2e^{-2t} \end{bmatrix} = \begin{bmatrix} 2e^t - 4e^{2t} + 2e^{3t} \\ -2e^{2t} + 2e^{3t} \end{bmatrix}$.

Finally, $\boldsymbol{y}(t) = \begin{bmatrix} -2e^{2t} + 3e^{3t} \\ -e^{2t} + 3e^{3t} \end{bmatrix} + \begin{bmatrix} 2e^t - 4e^{2t} + 2e^{3t} \\ -2e^{2t} + 2e^{3t} \end{bmatrix} = \begin{bmatrix} 2e^t - 6e^{2t} + 5e^{3t} \\ -3e^{2t} + 5e^{3t} \end{bmatrix}$.

Sage. Here is how we can let Sage do these computations for us:

```
>>> s, t = var('s, t')
>>> A = matrix([[1,2],[-1,4]])
>>> y = exp(A*t)*vector([1,2]) + exp(A*t)*integrate(exp(-A*s)*vector([0,2*e^s]), s,0,t)
>>> y.simplify_full()
```

$$(5\, e^{(3\,t)} - 6\, e^{(2\,t)} + 2\, e^t,\; 5\, e^{(3\,t)} - 3\, e^{(2\,t)})$$

Comment. Can you see that the solution is of the form that we anticipate from the method of undetermined coefficients?
Indeed, $\boldsymbol{y}(t) = \boldsymbol{y}_p(t) + \boldsymbol{y}_h(t)$ where the simplest particular solution is $\boldsymbol{y}_p(t) = \begin{bmatrix} 2e^t \\ 0 \end{bmatrix}$.

Modeling

Example 96. Consider two brine tanks. Tank T_1 contains 24gal water containing 3lb salt, and tank T_2 contains 9gal pure water.

- T_1 is being filled with 54gal/min water containing 0.5lb/gal salt.
- 72gal/min well-mixed solution flows out of T_1 into T_2.
- 18gal/min well-mixed solution flows out of T_2 into T_1.
- Finally, 54gal/min well-mixed solution is leaving T_2.

We wish to understand how much salt is in the tanks after t minutes.

(a) Derive a system of differential equations.

(b) Determine the equilibrium points and classify their stability. What does this mean here?

(c) Solve the system to find explicit formulas for how much salt is in the tanks after t minutes.

Solution.

(a) Note that the amount of water in each tank is constant because the flows balance each other.
Let $y_i(t)$ denote the amount of salt (in lb) in tank T_i after time t (in min). In time interval $[t, t+\Delta t]$:
$\Delta y_1 \approx 54 \cdot \frac{1}{2} \cdot \Delta t - 72 \cdot \frac{y_1}{24} \cdot \Delta t + 18 \cdot \frac{y_2}{9} \cdot \Delta t$, so $y_1' = 27 - 3y_1 + 2y_2$. Also, $y_1(0) = 3$.
$\Delta y_2 \approx 72 \cdot \frac{y_1}{24} \cdot \Delta t - 72 \cdot \frac{y_2}{9} \cdot \Delta t$, so $y_2' = 3y_1 - 8y_2$. Also, $y_2(0) = 0$.
Using matrix notation and writing $\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, this is $\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{y} = \begin{bmatrix} -3 & 2 \\ 3 & -8 \end{bmatrix}\boldsymbol{y} + \begin{bmatrix} 27 \\ 0 \end{bmatrix}$, $\boldsymbol{y}(0) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$.

(b) Note that this system is autonomous! Otherwise, we could not pursue our stability analysis.
To find the equilibrium point (since the system is linear, there should be just one), we set $\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{y} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and solve $\begin{bmatrix} -3 & 2 \\ 3 & -8 \end{bmatrix}\boldsymbol{y} + \begin{bmatrix} 27 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. We find $\boldsymbol{y} = \begin{bmatrix} -3 & 2 \\ 3 & -8 \end{bmatrix}^{-1}\begin{bmatrix} -27 \\ 0 \end{bmatrix} = \begin{bmatrix} 12 \\ 4.5 \end{bmatrix}$.
The characteristic polynomial of $\begin{bmatrix} -3 & 2 \\ 3 & -8 \end{bmatrix}$ is $(-3-\lambda)(-8-\lambda) - 6 = \lambda^2 + 11\lambda + 18 = (\lambda+9)(\lambda+2)$.
Hence, the eigenvalues are $-9, -2$. Since they are both negative, the equilibrium point is a nodal sink and, in particular, asymptotically stable.
Having an equilibrium point at $(12, 4.5)$, means that, if the salt amounts are $y_1 = 12$, $y_2 = 4.5$, then they won't change over time (but will remain unchanged at these levels). The fact that it is asymptotically stable means that salt amounts close to these balanced levels will, over time, approach the equilibrium levels. (Because the system is linear, this is also true for levels that are not "close".)
We could have "seen" the equilibrium point!
Indeed, noticing that, for a constant (equilibrium) particular solution $\boldsymbol{y}$, each tank has to have a constant concentration of 0.5lb/gal of salt, we find directly $\boldsymbol{y} = 0.5\begin{bmatrix} 24 \\ 9 \end{bmatrix} = \begin{bmatrix} 12 \\ 4.5 \end{bmatrix}$.

(c) This is an IVP that we can solve (with some work). Do it! For instance, we can apply variation of constants. (Alternatively, leverage our particular solution from the previous part!) Skipping most work, we find:

- If $A = \begin{bmatrix} -3 & 2 \\ 3 & -8 \end{bmatrix}$, then $e^{At} = \frac{1}{7}\begin{bmatrix} e^{-9t} + 6e^{-2t} & -2e^{-9t} + 2e^{-2t} \\ -3e^{-9t} + 3e^{-2t} & 6e^{-9t} + 1e^{-2t} \end{bmatrix}$
- $\boldsymbol{y} = e^{At}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{At}\int_0^t e^{-As}\begin{bmatrix} 27 \\ 0 \end{bmatrix}\mathrm{d}s = \frac{1}{7}\begin{bmatrix} e^{-9t} + 6e^{-2t} \\ -3e^{-9t} + 3e^{-2t} \end{bmatrix} + \frac{3}{14}e^{At}\begin{bmatrix} 2e^{9t} + 54e^{2t} - 56 \\ -6e^{9t} + 27e^{2t} - 21 \end{bmatrix}$
$= \begin{bmatrix} 12 - 9e^{-2t} \\ 4.5 - 4.5e^{-2t} \end{bmatrix}$

Lotka–Volterra predator–prey model

The Lotka–Volterra equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \alpha x - \beta x y, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \delta x y - \gamma y,$$

are used, for instance, in biology to describe the dynamics of two species that interact, one as a predator and the other as prey.

Can you put into words how these equations might indeed describe the interactions between predator and prey? (Here, $\alpha, \beta, \gamma, \delta$ are positive real constants.)

To begin with, which of x and y is the predator and which is the prey?

What are the equations saying about a population of only predator or only prey?

For more information: https://en.wikipedia.org/wiki/Lotka-Volterra_equations

Example 97. Determine the equilibrium points of the Lotka–Volterra equations and classify their stability. What does this mean for this problem?

Solution. Solving $\alpha x - \beta x y = x(\alpha - \beta y) = 0$ and $\delta x y - \gamma y = (\delta x - \gamma) y = 0$, we find that there are two equilibrium points: $(0, 0)$ and $\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$.

The Jacobian matrix of $\left[\begin{array}{c} f\\ g \end{array}\right] = \left[\begin{array}{c} \alpha x - \beta x y\\ \delta x y - \gamma y \end{array}\right]$ is $J = \left[\begin{array}{cc} f_x & f_y\\ g_x & g_y \end{array}\right] = \left[\begin{array}{cc} \alpha - \beta y & - \beta x\\ \delta y & \delta x - \gamma \end{array}\right]$.

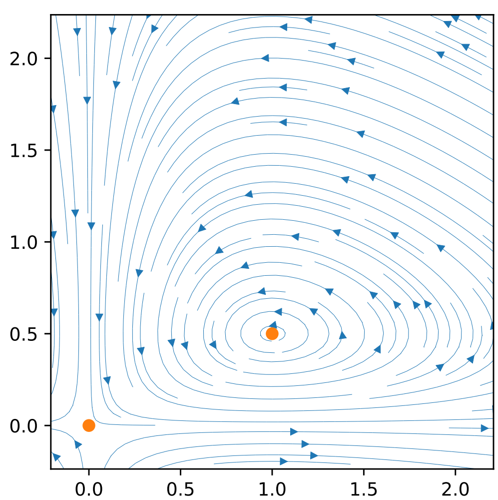
- At $(0, 0)$, the Jacobian matrix is $J = \left[\begin{array}{cc} \alpha & 0\\ 0 & - \gamma \end{array}\right]$. The eigenvalues are α and $- \gamma$.

  Since these are real with opposite signs, $(0, 0)$ ("extinction") is a saddle and, in particular, unstable.

- At $\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$, the Jacobian matrix is $J = \left[\begin{array}{cc} 0 & - \beta \gamma / \delta\\ \alpha \delta / \beta & 0 \end{array}\right]$. The characteristic polynomial is $\lambda^2 + \alpha \gamma$ so that the eigenvalues are $\pm i \sqrt{\alpha \gamma}$.

  Since the eigenvalues are pure imaginary, we cannot immediately predict stability (the equilibrium point of the linearization is a center but our equilibrium point could be either a center or a spiral source/sink).

  A closer inspection shows that the equilibrium point here is a center (see the comment below). This is confirmed by the following phase portrait for $\alpha = \frac{2}{3}$, $\beta = \frac{4}{3}$, $\gamma = \delta = 1$.



Armin Straub
straub@southalabama.edu

Comment. The equilibrium point $\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$ has the interesting feature that the stable population for y (the predator) depends on the growth parameters for x (the prey). For instance, increasing the birth rate α of the prey (for instance, by improving the environment for the prey) ends up benefitting the predator but not the prey! (See the wikipedia article for links with the "paradox of enrichment" and how such effects can indeed be observed in actual populations.)

Comment. Here is one way to conclude that $\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$ is a center and, therefore, stable.

We can eliminate t from the DEs to arrive at $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(\delta x - \gamma)y}{x(\alpha - \beta y)}$.

In general, solutions to this DE describe the trajectories in our phase plots.

Here, the DE for $\frac{\mathrm{d}y}{\mathrm{d}x}$ is separable: $\frac{\alpha - \beta y}{y}\mathrm{d}y = \frac{\delta x - \gamma}{x}\mathrm{d}x$. Integrating (and using that $x, y > 0$), we find that

$$\alpha \ln(y) - \beta y = \delta \ln(x) - \gamma x + C.$$

This means that the trajectories in our phase portrait are level curves of the function $\alpha \ln(y) - \beta y - \delta \ln(x) + \gamma x$. Since this function has no anomalies for $x, y > 0$, these level curves cannot be spiralling towards the equilibrium point (for instance, we can fix values for $x > 0$ and C, and then observe that $\alpha \ln(y) - \beta y = D$ with $D = \delta \ln(x) - \gamma x + C$ has at most two solutions for y and certainly not infinitely many). Thus, the equilibrium point is a center.

Bonus: Two more applications of systems of DEs

Example 98. (epidemiology) Let us indicate the popular SIR model for short outbreaks of diseases among a population of constant size N.

In a SIR model, the population is compartmentalized into $S(t)$ susceptible, $I(t)$ infected and $R(t)$ recovered (or resistant) individuals ($N = S(t) + I(t) + R(t)$). In the Kermack-McKendrick model, the outbreak of a disease is modeled by

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \gamma I, \quad \frac{\mathrm{d}S}{\mathrm{d}t} = -\beta S I, \quad \frac{\mathrm{d}I}{\mathrm{d}t} = \beta S I - \gamma I,$$

with γ modeling the recovery rate and β the infection rate. Note that this is a non-linear system of differential equations. For more details and many variations used in epidemiology, see:

`https://en.wikipedia.org/wiki/Compartmental_models_in_epidemiology`

Comment. The following variation

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \gamma I R, \quad \frac{\mathrm{d}S}{\mathrm{d}t} = -\beta S I, \quad \frac{\mathrm{d}I}{\mathrm{d}t} = \beta S I - \gamma I R,$$

which assumes "infectious recovery", was used in 2014 to predict that facebook might lose 80% of its users by 2017. It's that claim, not mathematics (or even the modeling), which attracted a lot of media attention.

`http://blogs.wsj.com/digits/2014/01/22/controversial-paper-predicts-facebook-decline/`

Example 99. (military strategy) Lanchester's equations model two opposing forces during "aimed fire" battle.

Let $x(t)$ and $y(t)$ describe the number of troops on each side. Then Lanchester (during World War I) assumed that the rates $-x'(t)$ and $-y'(t)$, at which soldiers are put out of action, are proportional to the number of opposing forces. That is:

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -\alpha y(t) \\ -\beta x(t) \end{bmatrix}, \quad \text{or, in matrix form: } \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -\alpha \\ -\beta & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

The proportionality constants $\alpha, \beta > 0$ indicate the strength of the forces ("fighting effectiveness coefficients").

These are simple linear DEs with constant coefficients, which we have learned how to solve.

For more details, see: `https://en.wikipedia.org/wiki/Lanchester%27s_laws`

Comment. The "aimed fire" means that all combatants are engaged, as is common in modern combat with long-range weapons. This is rather different than ancient combat where soldiers were engaging one opponent at a time.

Some special functions and the power series method

Review: power series

Definition 100. A function $y(x)$ is analytic around $x = x_0$ if it has a power series

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

Note. In the next theorem, we will see that this power series is the Taylor series of $y(x)$ around $x = x_0$.

Power series are very pleasant to work with because they behave just like polynomials. For instance, we can differentiate and integrate them:

- If $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$, then $y'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}$ (another power series!).

  We can rewrite the series as $y'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} (x - x_0)^n$.

  The result is a power series just like the one we started with. Likewise, for higher derivatives.

- $\int y(x) \mathrm{d}x = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x - x_0)^{n+1} + C$

Theorem 101. If $y(x)$ is analytic around $x = x_0$, then $y(x)$ is infinitely differentiable and

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad \text{with} \quad a_n = \frac{y^{(n)}(x_0)}{n!}.$$

Caution. Analyticity is needed in this theorem; being infinitely differentiable is not enough. For instance, $y(x) = e^{-1/x^2}$ is infinitely differentiable around $x = 0$ (and everywhere else). However, $y^{(n)}(0) = 0$ for all n.

In particular, if $y(x)$ is analytic at $x = 0$, then

$$y(x) = \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!} x^n = y(0) + y'(0)x + \frac{1}{2} y''(0) x^2 + \frac{1}{6} y'''(0) x^3 + \ldots$$

We have already seen the following example.

Example 102. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{1}{2} x^2 + \frac{1}{3!} x^3 + \ldots$

Once again, notice how the power series clearly has the property that $y' = y$ (as well as $y(0) = 1$).

It follows from here that, for instance, $e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = 1 + 2x + 2x^2 + \frac{4}{3} x^3 + \ldots$

Example 103. Determine the power series for $7e^{3x}$ (at $x = 0$).

Solution. Instead of starting from scratch, we can use that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ to conclude that

$$7e^{3x} = 7 \sum_{n=0}^{\infty} \frac{(3x)^n}{n!} = \sum_{n=0}^{\infty} \frac{7 \cdot 3^n}{n!} x^n = 7 + 21x + \frac{63}{2} x^2 + \frac{63}{2} x^3 + \frac{189}{8} x^4 + \ldots$$

Example 104. Determine the first several terms in the power series of $\sin(2x^3)$ at $x=0$.

Solution. (direct—unpleasant) If $f(x)=\sin(2x^3)$, then $f'(x)=6x^2\cos(2x^3)$ as well as $f''(x)=12x\cos(2x^3)-36x^4\sin(2x^3)$ and $f'''(x)=12\cos(2x^3)-216x^3\sin(2x^3)+216x^6\cos(2x^3)$.

In particular, $f(0)=0$, $f'(0)=0$, $f''(0)=0$ and $f'''(0)=12$.

It follows that $f(x)=f(0)+f'(0)x+\frac{1}{2}f''(0)x^2+\ldots=0+0x+0x^2+\frac{12}{3!}x^3+\ldots=2x^3+\ldots$

Solution. (via series for sine)

Example 105. Determine the power series for $\cos(x)$ at $x=0$.

Solution. (via DE) $\cos(x)$ is the unique solution to the IVP $y''=-y$, $y(0)=1$, $y'(0)=0$.

It follows that $\cos(x)=\sum_{n=0}^{\infty}a_nx^n$ with $a_n=\frac{y^{(n)}(0)}{n!}$. The DE implies that $y^{(2n)}(x)=(-1)^ny(x)$ and $y^{(2n+1)}=(-1)^ny'(x)$ so that $y^{(2n)}(0)=(-1)^n$ and $y^{(2n+1)}(0)=0$. Consequently, $\cos(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}$.

Solution. (via Euler's formula) Recall that $e^{ix}=\cos(x)+i\sin(x)$. Since

$e^{ix}=\sum_{n=0}^{\infty}\frac{(ix)^n}{n!}=\sum_{m=0}^{\infty}\frac{(ix)^{2m}}{(2m)!}+\sum_{m=0}^{\infty}\frac{(ix)^{2m+1}}{(2m+1)!}=\sum_{m=0}^{\infty}\frac{(-1)^mx^{2m}}{(2m)!}+i\sum_{m=0}^{\infty}\frac{(-1)^mx^{2m+1}}{(2m+1)!}$,

we conclude that $\cos(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}$ and $\sin(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$.

Example 106. Determine the first several terms in the power series of $\sin(2x^3)$ at $x=0$.

Solution. (direct—unpleasant) If $f(x)=\sin(2x^3)$, then $f'(x)=6x^2\cos(2x^3)$ as well as $f''(x)=12x\cos(2x^3)-36x^4\sin(2x^3)$ and $f'''(x)=12\cos(2x^3)-216x^3\sin(2x^3)+216x^6\cos(2x^3)$.

In particular, $f(0)=0$, $f'(0)=0$, $f''(0)=0$ and $f'''(0)=12$.

It follows that $f(x)=f(0)+f'(0)x+\frac{1}{2}f''(0)x^2+\ldots=0+0x+0x^2+\frac{12}{3!}x^3+\ldots=2x^3+\ldots$

Solution. (via series for sine) As derived in the previous example, we have

$$\sin(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}=x-\frac{1}{6}x^3+\frac{1}{120}x^5-\ldots$$

It follows that

$$\begin{aligned}\sin(2x^3) &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(2x^3)^{2n+1}=\sum_{n=0}^{\infty}\frac{(-1)^n2^{2n+1}}{(2n+1)!}x^{6n+3}\\ &= \frac{2^1}{1!}x^3-\frac{2^3}{3!}x^9+\frac{2^5}{5!}x^{15}-\ldots=2x^3-\frac{4}{3}x^9+\frac{4}{15}x^{15}-\ldots\end{aligned}$$

Review. Lotka–Volterra predator–prey model

Power series solutions to DE

Given any DE, we can approximate analytic solutions by working with the first few terms of the power series.

Example 107. (Airy equation, part I) Let $y(x)$ be the unique solution to the IVP $y'' = xy$, $y(0) = a$, $y'(0) = b$. Determine the first several terms (up to x^6) in the power series of $y(x)$.

Solution. (successive differentiation) From the DE, $y''(0) = 0 \cdot y(0) = 0$.

Differentiating both sides of the DE, we obtain $y''' = y + xy'$ so that $y'''(0) = y(0) + 0 \cdot y'(0) = a$.

Likewise, $y^{(4)} = 2y' + xy''$ shows $y^{(4)}(0) = 2y'(0) = 2b$.

Continuing, $y^{(5)} = 3y'' + xy'''$ so that $y^{(5)}(0) = 3y''(0) = 0$.

Continuing, $y^{(6)} = 4y''' + xy^{(4)}$ so that $y^{(6)}(0) = 4y'''(0) = 4a$.

Hence, $y(x) = a + bx + \frac{1}{2}y''(0)x^2 + \frac{1}{6}y'''(0)x^3 + \frac{1}{24}y^{(4)}(0)x^4 + \frac{1}{120}y^{(5)}(0)x^5 + \frac{1}{720}y^{(6)}(0)x^6 + \ldots$
$= a + bx + \frac{a}{6}x^3 + \frac{b}{12}x^4 + \frac{a}{180}x^6 + \ldots$

Comment. Do you see the general pattern? We will revisit this example soon.

Solution. (plug in power series) The powers series $y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots$ becomes $y = a + bx + a_2x^2 + a_3x^3 + a_4x^4 + \ldots$ because of the initial conditions.

To determine a_2, a_3, a_4, a_5, a_6, we equate the coefficients of:

$$\begin{aligned} y'' &= 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + \ldots \\ xy &= ax + bx^2 + a_2x^3 + a_3x^4 + \ldots \end{aligned}$$

We find $2a_2 = 0$, $6a_3 = a$, $12a_4 = b$, $20a_5 = a_2$, $30a_6 = a_3$.

So $a_2 = 0$, $a_3 = \frac{a}{6}$, $a_4 = \frac{b}{12}$, $a_5 = \frac{a_2}{20} = 0$, $a_6 = \frac{a_3}{30} = \frac{a}{180}$. Hence, $y(x) = a + bx + \frac{a}{6}x^3 + \frac{b}{12}x^4 + \frac{a}{180}x^6 + \ldots$

Armin Straub
straub@southalabama.edu

Review. If $y(x)$ is "nice" at $x=x_0$ (i.e. analytic around $x=x_0$), then

$$y(x)=\sum_{n=0}^{\infty} a_n(x-x_0)^n \quad \text{with} \quad a_n=\frac{y^{(n)}(x_0)}{n!}.$$

In particular, at $x=0$,

$$y(x)=\sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!} x^n = y(0)+y'(0)x+\frac{1}{2}y''(0)x^2+\frac{1}{6}y'''(0)x^3+\ldots$$

Notation. When working with power series $\sum_{n=0}^{\infty} a_n x^n$, we sometimes write $O(x^n)$ to indicate that we omit terms that are multiples of x^n:

For instance. $e^x=1+x+\frac{1}{2}x^2+O(x^3)$ or $\cos(x)=1-\frac{1}{2}x^2+\frac{1}{24}x^4+O(x^6)$.

Example 108. Let $y(x)$ be the unique solution to the IVP $y'=x^2+y^2$, $y(0)=1$. Determine the first several terms (up to x^4) in the power series of $y(x)$.

Solution. (successive differentiation) From the DE, $y'(0)=0^2+y(0)^2=1$.

Differentiating both sides of the DE, we obtain $y''=2x+2yy'$. In particular, $y''(0)=2$.

Continuing, $y'''=2+2(y')^2+2yy''$ so that $y'''(0)=2+2+2\cdot 2=8$.

Likewise, $y^{(4)}=6y'y''+2yy'''$ so that $y^{(4)}(0)=12+16=28$.

Hence, $y(x)=y(0)+y'(0)x+\frac{1}{2}y''(0)x^2+\frac{1}{6}y'''(0)x^3+\frac{1}{24}y^{(4)}(0)x^4+\ldots=1+x+x^2+\frac{4}{3}x^3+\frac{7}{6}x^4+\ldots$

Comment. This approach requires the (symbolic) computation of intermediate derivatives. This is costly (even just the size of the simplified formulas is quickly increasing) and so the solution below is usually preferable for practical purposes. However, successive differentiation works well when working by hand.

Solution. (plug in power series) The powers series $y=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\ldots$ simplifies to $y=1+a_1x+a_2x^2+a_3x^3+a_4x^4+\ldots$ because of the initial condition.

Therefore, $y'=a_1+2a_2x+3a_3x^2+4a_4x^3+\ldots$

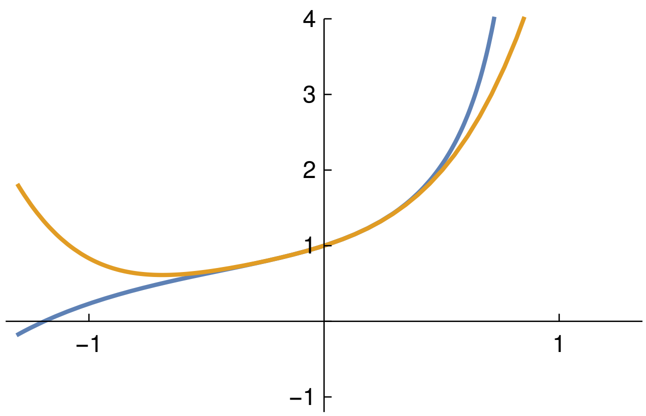
To determine a_2,a_3,a_4,a_5, we need to expand x^2+y^2 into a power series:

$y^2=1+2a_1x+(2a_2+a_1^2)x^2+(2a_3+2a_1a_2)x^3+(2a_4+2a_1a_3+a_2^2)x^4+\ldots$ [we don't need the last term]

Equating coefficients of y' and x^2+y^2, we find $a_1=1$, $2a_2=2a_1$, $3a_3=1+2a_2+a_1^2$, $4a_4=2a_3+2a_1a_2$.

So $a_1=1$, $a_2=1$, $a_3=\frac{4}{3}$, $a_4=\frac{7}{6}$ and, hence, $y(x)=1+x+x^2+\frac{4}{3}x^3+\frac{7}{6}x^4+\ldots$

Below is a plot of $y(x)$ (in blue) and our approximation:



Note how the approximation is very good close to 0 but does not provide us with a "global picture".

Armin Straub
straub@southalabama.edu

Example 109. Let $y(x)$ be the unique solution to the IVP $y''=\cos(x+y)$, $y(0)=0$, $y'(0)=1$. Determine the first several terms (up to x^5) in the power series of $y(x)$.

Solution. (successive differentiation) From the DE, $y''(0)=\cos(0+y(0))=1$.
Differentiating both sides of the DE, we obtain $y'''=-\sin(x+y)(1+y')$.
In particular, $y'''(0)=-\sin(0+y(0))(1+y'(0))=0$.
Likewise, $y^{(4)}=-\cos(x+y)(1+y')^2-\sin(x+y)y''$ shows $y^{(4)}(0)=-1\cdot 2^2-0=-4$.
Continuing, $y^{(5)}=\sin(x+y)(1+y')^3-3\cos(x+y)(1+y')y''-\sin(x+y)y'''$ so that $y^{(5)}(0)=-6$.
Hence, $y(x)=x+\frac{1}{2}y''(0)x^2+\frac{1}{6}y'''(0)x^3+\frac{1}{24}y^{(4)}(0)x^4+\frac{1}{120}y^{(5)}(0)x^5+\ldots=x+\frac{1}{2}x^2-\frac{1}{6}x^4-\frac{1}{20}x^5+\ldots$

Solution. (plug in power series) The powers series $y=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\ldots$ simplifies to $y=x+a_2x^2+a_3x^3+a_4x^4+\ldots$ because of the initial conditions.
Therefore, $y'=1+2a_2x+3a_3x^2+4a_4x^3+\ldots$ and $y''=2a_2+6a_3x+12a_4x^2+20a_5x^3+\ldots$
To determine a_2,a_3,a_4,a_5, we need to expand $\cos(x+y)$ into a power series:

Recall that $\cos(x)=1-\frac{1}{2}x^2+\frac{1}{24}x^4+\ldots$

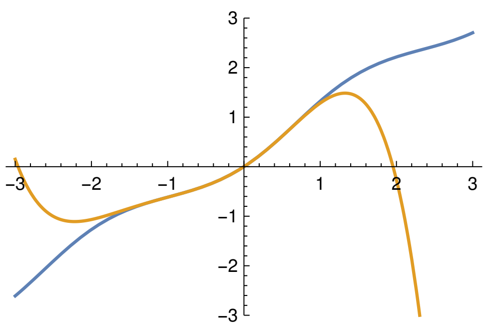
Hence, $\cos(x+y)=1-\frac{1}{2}(x+y)^2+\frac{1}{24}(x+y)^4+\ldots=1-\frac{1}{2}x^2-xy-\frac{1}{2}y^2+O(x^4)$.

Since $y^2=(x+a_2x^2+a_3x^3+\ldots)^2=x^2+2a_2x^3+O(x^4)$,

$\cos(x+y)=1-\frac{1}{2}x^2-x(x+a_2x^2)-\frac{1}{2}(x^2+2a_2x^3)+O(x^4)=1-2x^2-2a_2x^3+O(x^4)$.

Equating coefficients of y'' and $\cos(x+y)$, we find $2a_2=1$, $6a_3=0$, $12a_4=-2$, $20a_5=-2a_2$.
So $a_2=\frac{1}{2}$, $a_3=0$, $a_4=-\frac{1}{6}$, $a_5=-\frac{1}{20}$ and, hence, $y(x)=x+\frac{1}{2}x^2-\frac{1}{6}x^4-\frac{1}{20}x^5+\ldots$

Below is a plot of $y(x)$ (in blue) and our approximation:



Armin Straub
straub@southalabama.edu

Power series solutions to linear DEs

Note how in the last two examples the "plug in power series" approach was complicated by the fact that the DE was not linear (we had to expand y^2 as well as $\cos(x+y)$, respectively).

For linear DEs, this complication does not arise and we can readily determine the complete power series expansion of analytic solutions (with a recursive description of the coefficients).

Example 110. (Airy equation, part II) Let $y(x)$ be the unique solution to the IVP $y'' = xy$, $y(0) = a$, $y'(0) = b$. Determine the power series of $y(x)$.

Solution. (plug in power series) Let us spell out the power series for y, y', y'' and xy:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$xy(x) = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1} x^n$$

Hence, $y'' = xy$ becomes $\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n = \sum_{n=1}^{\infty} a_{n-1} x^n$. We compare coefficients of x^n:

- $n = 0$: $2 \cdot 1 a_2 = 0$, so that $a_2 = 0$.
- $n \geqslant 1$: $(n+2)(n+1) a_{n+2} = a_{n-1}$
  Replacing n by $n-2$, this is equivalent to $n(n-1) a_n = a_{n-3}$ for $n \geqslant 3$.

In conclusion, $y(x) = \sum_{n=0}^{\infty} a_n x^n$ with $a_0 = a$, $a_1 = b$, $a_2 = 0$ as well as, for $n \geqslant 3$, $a_n = \frac{1}{n(n-1)} a_{n-3}$.

First few terms. In particular, $y = a\left(1 + \frac{x^3}{2 \cdot 3} + \frac{x^6}{(2 \cdot 3)(5 \cdot 6)} + \ldots\right) + b\left(x + \frac{x^4}{3 \cdot 4} + \frac{x^7}{(3 \cdot 4)(6 \cdot 7)} + \ldots\right)$.

Advanced. The solution with $y(0) = \frac{1}{3^{2/3}\Gamma(2/3)}$ and $y'(0) = -\frac{1}{3^{1/3}\Gamma(1/3)}$ is known as the **Airy function** $\mathrm{Ai}(x)$.
[A more natural property of $\mathrm{Ai}(x)$ is that it satisfies $y(x) \to 0$ as $x \to \infty$.]

Once we have a power series solution $y(x)$, a natural question is: for which x does the series converge?

Recall. A power series $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ has a **radius of convergence** R.
The series converges for all x with $|x - x_0| < R$ and it diverges for all x with $|x - x_0| > R$.

Definition 111. Consider the linear DE $y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \ldots + p_1(x) y' + p_0(x) y = f(x)$.
x_0 is called an **ordinary point** if the coefficients $p_j(x)$, as well as $f(x)$, are analytic at $x = x_0$.
Otherwise, x_0 is called a **singular point**.

Example 112. Determine the singular points of $(x+2) y'' - x^2 y + 3y = 0$.

Solution. Rewriting the DE as $y'' - \frac{x^2}{x+2} y + \frac{3}{x+2} y = 0$, we see that the only singular point is $x = -2$.

Example 113. Determine the singular points of $(x^2+1) y''' = \frac{y}{x-5}$.

Solution. Rewriting the DE as $y''' - \frac{1}{(x-5)(x^2+1)} y = 0$, we see that the singular points are $x = \pm i, 5$.

Theorem 114. Consider the linear DE $y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \ldots + p_1(x) y' + p_0(x) y = f(x)$.
Suppose that x_0 is an ordinary point and that R is the distance to the closest singular point.
Then any IVP specifying $y(x_0)$, $y'(x_0)$, ..., $y^{(n-1)}(x_0)$ has a power series solution $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ and that series has radius of convergence at least R.

In particular. The DE has a general solution consisting of n solutions $y(x)$ that are analytic at $x = x_0$.

Comment. Most textbooks only discuss the case of 2nd order DEs. For a discussion of the higher order case (in terms of first order systems!) see, for instance, Chapter 4.5 in *Ordinary Differential Equations* by N. Lebovitz.
The book is freely available at: `http://people.cs.uchicago.edu/~lebovitz/odes.html`

Example 115. Find a minimum value for the radius of convergence of a power series solution to $(x+2) y'' - x^2 y' + 3y = 0$ at $x = 3$.

Solution. As before, rewriting the DE as $y'' - \frac{x^2}{x+2} y' + \frac{3}{x+2} y = 0$, we see that the only singular point is $x = -2$.
Note that $x = 3$ is an ordinary point of the DE and that the distance to the singular point is $|3 - (-2)| = 5$.
Hence, the DE has power series solutions about $x = 3$ with radius of convergence at least 5.

Example 116. Find a minimum value for the radius of convergence of a power series solution to $(x^2+1) y''' = \frac{y}{x-5}$ at $x = 2$.

Solution. As before, rewriting the DE as $y''' - \frac{1}{(x-5)(x^2+1)} y = 0$, we see that the singular points are $x = \pm i, 5$.
Note that $x = 2$ is an ordinary point of the DE and that the distance to the nearest singular point is $|2 - i| = \sqrt{5}$ (the distances are $|2 - 5| = 3$, $|2 - i| = |2 - (-i)| = \sqrt{2^2 + 1^2} = \sqrt{5}$).
Hence, the DE has power series solutions about $x = 2$ with radius of convergence at least $\sqrt{5}$.

Example 117. **(Airy equation, once more)** Let $y(x)$ be the solution to the IVP $y'' = xy$, $y(0) = a$, $y'(0) = b$. Earlier, we determined the power series of $y(x)$. What is its radius of convergence?

Solution. $y'' = xy$ has no singular points. Hence, the radius of convergence is ∞. (In other words, the power series converges for all x.)

Review. Theorem 114: If x_0 is an ordinary point of a linear IVP, then it is guaranteed to have a power series solution $y(x)=\sum_{n=0}^{\infty}a_n(x-x_0)^n$.

Moreover, its radius of convergence is at least the distance between x_0 and the closest singular point.

Example 118. Find a minimum value for the radius of convergence of a power series solution to $(x^2+4)y''-3xy'+\frac{1}{x+1}y=0$ at $x=2$.

Solution. The singular points are $x=\pm 2i,-1$. Hence, $x=2$ is an ordinary point of the DE and the distance to the nearest singular point is $|2-2i|=\sqrt{2^2+2^2}=\sqrt{8}$ (the distances are $|2-(-1)|=3$, $|2-(\pm 2i)|=\sqrt{8}$). By Theorem 114, the DE has power series solutions about $x=2$ with radius of convergence at least $\sqrt{8}$.

Example 119. (caution!) Theorem 114 only holds for linear DEs! For nonlinear DEs, it is very hard to predict whether there is a power series solution and what its radius of convergence is. Consider, for instance, the nonlinear DE $y'-y^2=0$.

Its coefficients have no singularities. A solution to this DE is $y(x)=\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$ (see Example 122), which clearly has a problem at $x=1$ (the radius of convergence is 1).

On the other hand. $y(x)$ also solves the linear DE $(1-x)y'-y=0$ (or, even simpler, the order 0 "differential" equation $(1-x)y=1$). Note how the DE has the singular point $x=1$. Theorem 114 then allows us to predict that $y(x)$ must have a power series with radius of convergence at least 1.

Example 120. (Bessel functions) Consider the DE $x^2y''+xy'+x^2y=0$. Derive a recursive description of a power series solutions $y(x)$ at $x=0$.

Caution! Note that $x=0$ is a singular point (the only) of the DE. Theorem 114 therefore does not guarantee a basis of power series solutions. [However, $x=0$ is what is called a **regular singular point**; for these, we are guaranteed one power series solution, as well as additional solutions expressed using logarithms and power series.]

Comment. We could divide the DE by x (but that wouldn't really change the computations below). The reason for not dividing that x is that this DE is the special case $\alpha=0$ of the **Bessel equation** $x^2y''+xy'+(x^2-\alpha^2)y=0$ (for which no such dividing is possible).

Solution. (plug in power series) Let us spell out power series for x^2y, xy', x^2y'' starting with $y(x)=\sum_{n=0}^{\infty}a_nx^n$:

$$x^2y(x)=\sum_{n=0}^{\infty}a_nx^{n+2}=\sum_{n=2}^{\infty}a_{n-2}x^n$$

$$xy'(x)=\sum_{n=1}^{\infty}na_nx^n \qquad \left(\text{because } y'(x)=\sum_{n=1}^{\infty}na_nx^{n-1}\right)$$

$$x^2y''(x)=\sum_{n=2}^{\infty}n(n-1)a_nx^n \qquad \left(\text{because } y''(x)=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}\right)$$

Hence, the DE becomes $\sum_{n=2}^{\infty}n(n-1)a_nx^n+\sum_{n=1}^{\infty}na_nx^n+\sum_{n=2}^{\infty}a_{n-2}x^n=0$. We compare coefficients of x^n:

- $n=1$: $\quad a_1=0$
- $n\geqslant 2$: $\quad n(n-1)a_n+na_n+a_{n-2}=0$, which simplifies to $n^2a_n=-a_{n-2}$.
  It follows that $a_{2n}=\frac{(-1)^n}{4^nn!^2}a_0$ and $a_{2n+1}=0$.

Observation. The fact that we found $a_1=0$ reflects the fact that we cannot represent the general solution through power series alone.

Comment. If $a_0=1$, the function we found is a **Bessel function** and denoted as $J_0(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!^2}\left(\frac{x}{2}\right)^{2n}$.

The more general Bessel functions $J_\alpha(x)$ are solutions to the DE $x^2y''+xy'+(x^2-\alpha^2)y=0$.

Armin Straub
straub@southalabama.edu

Example 121. (caution!) Consider the linear DE $x^2 y' = y - x$. Does it have a convergent power series solution at $x = 0$?

Important note. The DE $x^2 y' = y - x$ has the singular point $x = 0$. Hence, Theorem 114 does not apply.

Advanced. Moreover, in contrast to the previous example, $x = 0$ is not a **regular singular point**. Indeed, as we see below, there is no power series solution of the DE at all.

Solution. Let us look for a power series solution $y(x) = \sum_{n=0}^{\infty} a_n x^n$.

$x^2 y'(x) = x^2 \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n+1} = \sum_{n=2}^{\infty} (n-1) a_{n-1} x^n$

Hence, $x^2 y' = y - x$ becomes $\sum_{n=2}^{\infty} (n-1) a_{n-1} x^n = \sum_{n=0}^{\infty} a_n x^n - x$. We compare coefficients of x^n:

- $n = 0$: $a_0 = 0$.
- $n = 1$: $0 = a_1 - 1$, so that $a_1 = 1$.
- $n \geqslant 2$: $(n-1) a_{n-1} = a_n$, from which it follows that $a_n = (n-1) a_{n-1} = (n-1)(n-2) a_{n-2} = \cdots = (n-1)! a_1 = (n-1)!$.

Hence the DE has the "formal" power series solution $y(x) = \sum_{n=1}^{\infty} (n-1)! x^n$.

However, that series is divergent for all $x \neq 0$; that is, the radius of convergence is 0.

Inverses of power series

Example 122. (geometric series) $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

Why? If $y(x) = \sum_{n=0}^{\infty} x^n$, then $xy = y - 1$ (write down the power series for both sides!). Hence, $y = \frac{1}{1-x}$.

Alternatively, start with $y = \frac{1}{1-x}$ and note that y solves the order 0 "differential" (inhomogeneous) equation $(1-x) y = 1$. We can then determine a power series solution as we did in Example 110 to find $y = \sum_{n=0}^{\infty} x^n$.

Example 123. (extra) Determine a power series for $\ln(x)$ around $x = 1$.

Solution. This is equivalent to finding a power series for $\ln(x+1)$ around $x = 0$ (see the final step).

Observe that $\ln(x+1) = \int \frac{\mathrm{d}x}{1+x} + C$ and that $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$.

Integrating, $\ln(x+1) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C$. Since $\ln(1) = 0$, we conclude that $C = 0$.

Finally, $\ln(x+1) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$ is equivalent to $\ln(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} (x-1)^{n+1}$.

Comment. Choosing $x = 2$ in $\ln(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} (x-1)^{n+1}$ results in $\ln(2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$.

The latter is the alternating harmonic sum.

Can you see from the series for $\ln(x)$ why the harmonic sum $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ diverges?

Armin Straub
straub@southalabama.edu

Example 124. Derive a recursive description of the power series for $y(x)=\frac{1}{1-x-x^2}$.

Solution. Note that $y(x)$ satisfies the "differential" equation $(1-x-x^2)y=1$ of order 0 (as such, we need 0 initial conditions). We can therefore determine a power series solution as we did in Example 110:

Write $y(x)=\sum_{n=0}^{\infty} a_n x^n$. Then

$$\begin{aligned} 1=(1-x-x^2)\sum_{n=0}^{\infty} a_n x^n &= \sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+2} \\ &= \sum_{n=0}^{\infty} a_n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n - \sum_{n=2}^{\infty} a_{n-2} x^n. \end{aligned}$$

We compare coefficients of x^n:

- $n=0$: $1=a_0$.
- $n=1$: $0=a_1-a_0$, so that $a_1=a_0=1$.
- $n\geqslant 2$: $0=a_n-a_{n-1}-a_{n-2}$ or, equivalently, $a_n=a_{n-1}+a_{n-2}$.

This is the recursive description of the Fibonacci numbers F_n! In particular $a_n=F_n$.

The first few terms. $\frac{1}{1-x-x^2}=1+x+2x^2+3x^3+5x^4+8x^5+13x^6+\ldots$

Comment. The function $y(x)$ is said to be a **generating function** for the Fibonacci numbers.

Challenge. Can you rederive Binet's formula from partial fractions and the geometric series?

Example 125. (HW) Derive a recursive description of the power series for $y(x)=\frac{1+7x}{1-x-2x^2}$.

Solution. Write $y(x)=\sum_{n=0}^{\infty} a_n x^n$. Then

$$\begin{aligned} 1+7x=(1-x-2x^2)\sum_{n=0}^{\infty} a_n x^n &= \sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+1} - 2\sum_{n=0}^{\infty} a_n x^{n+2} \\ &= \sum_{n=0}^{\infty} a_n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n - 2\sum_{n=2}^{\infty} a_{n-2} x^n. \end{aligned}$$

We compare coefficients of x^n:

- $n=0$: $1=a_0$.
- $n=1$: $7=a_1-a_0$, so that $a_1=7+a_0=8$.
- $n\geqslant 2$: $0=a_n-a_{n-1}-2a_{n-2}$.

If we prefer, we can rewrite the final recurrence as $a_{n+2}-a_{n+1}-2a_n=0$ for $n\geqslant 0$. The initial conditions are $a_0=1$, $a_1=8$.

Comment. In terms of the recurrence operator N, the recurrence is $(N^2-N-2)a_n=0$.

Comment. As in Example 55, we can solve this recurrence and obtain a Binet-like formula for a_n. In this particular case, we find $a_n=3\cdot 2^n-2(-1)^n$.

Armin Straub
straub@southalabama.edu

Power series of familiar functions

(Unless we specify otherwise, power series are meant to be about $x=0$.)

Example 126. The **hyperbolic cosine** $\cosh(x)$ is defined to be the even part of e^x. In other words, $\cosh(x)=\frac{1}{2}(e^x+e^{-x})$. Determine its power series.

Solution. It follows from $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$ that $\cosh(x)=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$.

Comment. Note that $\cosh(ix)=\cos(x)$ (because $\cos(x)=\frac{1}{2}(e^{ix}+e^{-ix})$).

Comment. The hyperbolic sine $\sinh(x)=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}$ is similarly defined to be the odd part of e^x.

Example 127. Determine a power series for $\frac{1}{1+x^2}$.

Solution. Replace x with $-x^2$ in $\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$ (geometric series!) to get $\frac{1}{1+x^2}=\sum_{n=0}^{\infty}(-1)^n x^{2n}$.

Example 128. Determine a power series for $\arctan(x)$.

Solution. Recall that $\arctan(x)=\int\frac{\mathrm{d}x}{1+x^2}+C$. Hence, we need to integrate $\frac{1}{1+x^2}=\sum_{n=0}^{\infty}(-1)^n x^{2n}$.

It follows that $\arctan(x)=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}+C$. Since $\arctan(0)=0$, we conclude that $C=0$.

Example 129. (error function) Determine a power series for $\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\mathrm{d}t$.

Solution. It follows from $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$ that $e^{-t^2}=\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{n!}$.

Integrating, we obtain $\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\mathrm{d}t=\frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{n!(2n+1)}$.

Example 130. Determine the first several terms (up to x^5) in the power series of $\tan(x)$.

Solution. Observe that $y(x)=\tan(x)$ is the unique solution to the IVP $y'=1+y^2$, $y(0)=0$.

We can therefore proceed to determine the first few power series coefficients as we did earlier.

That is, we plug $y=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\ldots$ into the DE. Note that $y(0)=0$ means $a_0=0$.

$y'=a_1+2a_2x+3a_3x^2+4a_4x^3+5a_5x^4+\ldots$

$1+y^2=1+(a_1x+a_2x^2+a_3x^3+\ldots)^2=1+a_1^2x^2+(2a_1a_2)x^3+(2a_1a_3+a_2^2)x^4+\ldots$

Comparing coefficients, we find: $a_1=1$, $2a_2=0$, $3a_3=a_1^2$, $4a_4=2a_1a_2$, $5a_5=2a_1a_3+a_2^2$.

Solving for $a_2,a_3,\ldots$, we conclude that $\tan(x)=x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}+\ldots$

Comment. The fact that $\tan(x)$ is an odd function translates into $a_n=0$ when n is even. If we had realized that at the beginning, our computation would have been simplified.

Advanced comment. The full power series is $\tan(x)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_{2n}}{(2n)!}x^{2n-1}$.

Here, the numbers B_{2n} are (rather mysterious) rational numbers known as **Bernoulli numbers**.

The radius of convergence is $\pi/2$. Note that this is not at all obvious from the DE $y'=1+y^2$. This illustrates the fact that nonlinear DEs are much more complicated than linear ones. (There is no analog of Theorem 114.)

Fourier series

The following amazing fact is saying that any 2π-periodic function can be written as a sum of cosines and sines.

Advertisement. In Linear Algebra II, we will see the following natural way to look at Fourier series: the functions 1, $\cos(t)$, $\sin(t)$, $\cos(2t)$, $\sin(2t)$, $\ldots$ are orthogonal to each other (for that to make sense, we need to think of functions as vectors and introduce a natural inner product). In fact, they form an orthogonal basis for the space of piecewise smooth functions. In that setting, the formulas for the coefficients a_n and b_n are nothing but the usual projection formulas for orthogonal projection onto a single vector.

Theorem 131. Every* 2π-periodic function f can be written as a **Fourier series**

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nt) + b_n \sin(nt)).$$

Technical detail*: f needs to be, e.g., piecewise smooth.

Also, if t is a discontinuity of f, then the Fourier series converges to the average $\frac{f(t^-) + f(t^+)}{2}$.

The **Fourier coefficients** a_n, b_n are unique and can be computed as

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) \mathrm{d}t, \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) \mathrm{d}t.$$

Comment. Another common way to write Fourier series is $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$.

These two ways are equivalent; we can convert between them using Euler's identity $e^{int} = \cos(nt) + i\sin(nt)$.

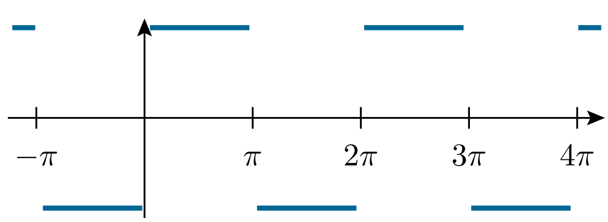
Definition 132. Let $L > 0$. $f(t)$ is ***L*-periodic** if $f(t + L) = f(t)$ for all t. The smallest such L is called the **(fundamental) period** of f.

Example 133. The fundamental period of $\cos(nt)$ is $2\pi / n$.

Example 134. The trigonometric functions $\cos(nt)$ and $\sin(nt)$ are 2π-periodic for every integer n. And so are their linear combinations. (Thus, 2π-periodic functions form a vector space!)

Example 135. Find the Fourier series of the 2π-periodic function $f(t)$ defined by

$$f(t) = \begin{cases} -1, & \text{for } t \in (-\pi, 0), \\ +1, & \text{for } t \in (0, \pi), \\ 0, & \text{for } t = -\pi, 0, \pi. \end{cases}$$

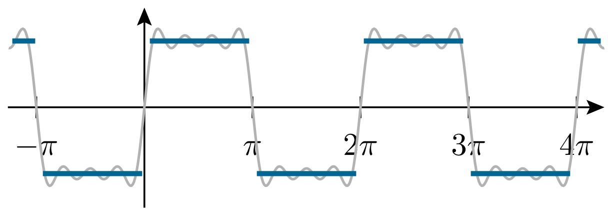


Armin Straub
straub@southalabama.edu

Solution. We compute $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\mathrm{d}t = 0$, as well as

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\mathrm{d}t = \frac{1}{\pi}\left[-\int_{-\pi}^{0}\cos(nt)\mathrm{d}t + \int_{0}^{\pi}\cos(nt)\mathrm{d}t\right] = 0 \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\mathrm{d}t = \frac{1}{\pi}\left[-\int_{-\pi}^{0}\sin(nt)\mathrm{d}t + \int_{0}^{\pi}\sin(nt)\mathrm{d}t\right] = \frac{2}{\pi n}[1-\cos(n\pi)] \\ &= \frac{2}{\pi n}[1-(-1)^n] = \begin{cases} \frac{4}{\pi n} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}. \end{aligned}$$

In conclusion, $f(t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n}\sin(nt) = \frac{4}{\pi}\left(\sin(t) + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t) + \ldots\right)$.



Observation. The coefficients a_n are zero for all n if and only if $f(t)$ is odd.

Comment. The value of $f(t)$ for $t = -\pi, 0, \pi$ is irrelevant to the computation of the Fourier series. They are chosen so that $f(t)$ is equal to the Fourier series for all t (recall that, at a jump discontinuity t, the Fourier series converges to the average $\frac{f(t^-)+f(t^+)}{2}$).

Comment. Plot the (sum of the) first few terms of the Fourier series. What do you observe? The "overshooting" is known as the **Gibbs phenomenon**: https://en.wikipedia.org/wiki/Gibbs_phenomenon

Comment. Set $t = \frac{\pi}{2}$ in the Fourier series we just computed, to get Leibniz' series $\pi = 4[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots]$. For such an alternating series, the error made by stopping at the term $1/n$ is on the order of $1/n$. To compute the 768 digits of π to get to the Feynman point ($3.14159265\ldots721134999999\ldots$), we would (roughly) need $1/n < 10^{-768}$, or $n > 10^{768}$. That's a lot of terms! (Roger Penrose, for instance, estimates that there are about 10^{80} atoms in the observable universe.)

Remark. Convergence of such series is not completely obvious! Recall, for instance, that the (odd part of) the harmonic series $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$ diverges. (On the other hand, do you remember the alternating sign test from Calculus II?)

Fourier series with general period

There is nothing special about 2π-periodic functions considered before (except that $\cos(t)$ and $\sin(t)$ have fundamental period 2π). Note that $\cos(\pi t/L)$ and $\sin(\pi t/L)$ have period $2L$.

Theorem 136. Every* $2L$-periodic function f can be written as a **Fourier series**

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right)\right).$$

Technical detail*: f needs to be, e.g., piecewise smooth.

Also, if t is a discontinuity, then the Fourier series converges to the average $\frac{f(t^-) + f(t^+)}{2}$.

The **Fourier coefficients** a_n, b_n are unique and can be computed as

$$a_n = \frac{1}{L} \int_{-L}^{L} f(t) \cos\left(\frac{n\pi t}{L}\right) \mathrm{d}t, \qquad b_n = \frac{1}{L} \int_{-L}^{L} f(t) \sin\left(\frac{n\pi t}{L}\right) \mathrm{d}t.$$

Comment. This follows from Theorem 131 because, if $f(t)$ has period $2L$, then $\tilde{f}(t) := f\left(\frac{L}{\pi} t\right)$ has period 2π.

Example 137. Find the Fourier series of the 2-periodic function $g(t) = \begin{cases} -1 & \text{for } t \in (-1, 0) \\ +1 & \text{for } t \in (0, 1) \\ 0 & \text{for } t = -1, 0, 1 \end{cases}$.

Solution. Instead of computing from scratch, we can use the fact that $g(t) = f(\pi t)$, with f as in the previous example, to get $g(t) = f(\pi t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \sin(n\pi t)$.

Theorem 138. If $f(t)$ is **continuous** and $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right)\right)$, then* $f'(t) = \sum_{n=1}^{\infty} \left(\frac{n\pi}{L} b_n \cos\left(\frac{n\pi t}{L}\right) - \frac{n\pi}{L} a_n \sin\left(\frac{n\pi t}{L}\right)\right)$ (i.e., we can differentiate termwise).

Technical detail*: f' needs to be, e.g., piecewise smooth (so that it has a Fourier series itself).

Caution! We cannot simply differentiate termwise if $f(t)$ is lacking continuity. See the next example.

Comment. On the other hand, we can integrate termwise (going from the Fourier series of $f' = g$ to the Fourier series of $f = \int g$ because the latter will be continuous). This is illustrated in the example after the next.

Example 139. (caution!) The function $g(t) = \sum_{n \text{ odd}} \frac{4}{\pi n} \sin(n\pi t)$ from Example 137 is not continuous. For all values, except the discontinuities, we have $g'(t) = 0$. On the other hand, differentiating the Fourier series termwise, results in $4 \sum_{n \text{ odd}} \cos(n\pi t)$, which diverges for most values of t (that's easy to check for $t = 0$). This illustrates that we cannot apply Theorem 138 because $g(t)$ is lacking continuity.

[The issues we are facing here can be fixed by generalizing the notion of function to distributions. (Maybe you have heard of the Dirac delta "function".)]

Armin Straub
straub@southalabama.edu

Example 140. Let $h(t)$ be the 2-periodic function with $h(t) = |t|$ for $t \in [-1, 1]$. Compute the Fourier series of $h(t)$.

Solution. We could just use the integral formulas to compute a_n and b_n. Since $h(t)$ is even (plot it!), we will find that $b_n = 0$. Computing a_n is left as an exercise.

Solution. Note that $h(t) = \begin{cases} -t & \text{for } t \in (-1, 0) \\ +t & \text{for } t \in (0, 1) \end{cases}$ is continuous and $h'(t) = g(t)$, with $g(t)$ as in Example 137. Hence, we can apply Theorem 138 to conclude

$$h'(t) = g(t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \sin(n\pi t) \quad \Longrightarrow \quad h(t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n}\left(-\frac{1}{\pi n}\right) \cos(n\pi t) + C,$$

where $C = \frac{a_0}{2} = \frac{1}{2}\int_{-1}^{1} h(t) \mathrm{d}t = \frac{1}{2}$ is the constant of integration. Thus, $h(t) = \frac{1}{2} - \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi^2 n^2} \cos(n\pi t)$.

Remark. Note that $t = 0$ in the last Fourier series, gives us $\frac{\pi^2}{8} = \frac{1}{1} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$. As an exercise, you can try to find from here the fact that $\sum_{n \geqslant 1} \frac{1}{n^2} = \frac{\pi^2}{6}$. Similarly, we can use Fourier series to find that $\sum_{n \geqslant 1} \frac{1}{n^4} = \frac{\pi^4}{90}$.
Just for fun. These are the values $\zeta(2)$ and $\zeta(4)$ of the Riemann zeta function $\zeta(s)$. No such evaluations are known for $\zeta(3), \zeta(5), \ldots$ and we don't even know (for sure) whether these are rational numbers. Nobody believes these to be rational numbers, but it was only in 1978 that Apéry proved that $\zeta(3)$ is not a rational number.

Sketch of Lecture 29

Mon, 11/6/2023

Fourier cosine series and Fourier sine series

Suppose we have a function $f(t)$ which is defined on a finite interval $[0, L]$. Depending on the kind of application, we can extend $f(t)$ to a periodic function in three natural ways; in each case, we can then compute a Fourier series for $f(t)$ (which will agree with $f(t)$ on $[0, L]$).

Comment. Here, we do not worry about the definition of $f(t)$ at specific individual points like $t = 0$ and $t = L$, or at jump discontinuities. Recall that, at a discontinuity, a Fourier series takes the average value.

(a) We can extend $f(t)$ to an L-periodic function.

In that case, we obtain the Fourier series $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{2\pi n t}{L}\right) + b_n \sin\left(\frac{2\pi n t}{L}\right)\right)$.

(b) We can extend $f(t)$ to an even $2L$-periodic function.

In that case, we obtain the **Fourier cosine series** $f(t) = \frac{\tilde{a}_0}{2} + \sum_{n=1}^{\infty} \tilde{a}_n \cos\left(\frac{\pi n t}{L}\right)$.

(c) We can extend $f(t)$ to an odd $2L$-periodic function.

In that case, we obtain the **Fourier sine series** $f(t) = \sum_{n=1}^{\infty} \tilde{b}_n \sin\left(\frac{\pi n t}{L}\right)$.

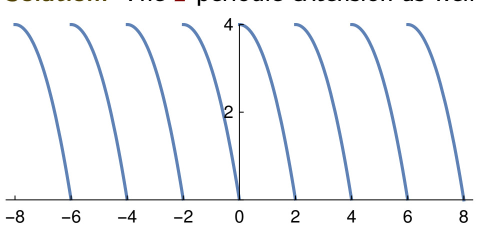
Example 141. Consider the function $f(t) = 4 - t^2$, defined for $t \in [0, 2]$.

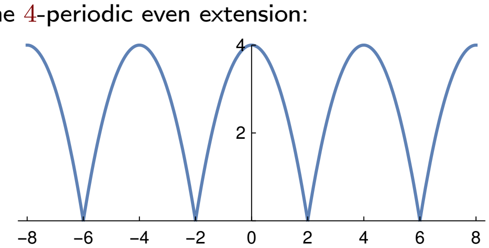
(a) Sketch the 2-periodic extension of $f(t)$.

(b) Sketch the 4-periodic even extension of $f(t)$.

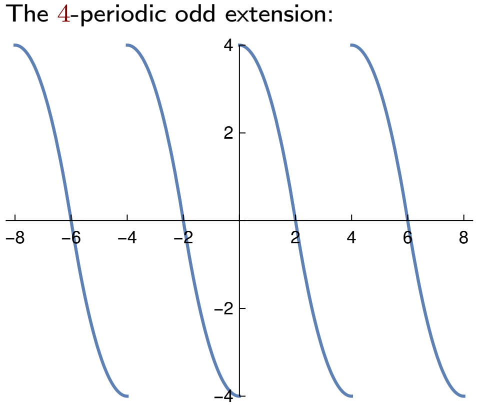
(c) Sketch the 4-periodic odd extension of $f(t)$.

Solution. The 2-periodic extension as well as the 4-periodic even extension:





The 4-periodic odd extension:



Armin Straub
straub@southalabama.edu

Example 142. As in the previous example, consider the function $f(t) = 4 - t^2$, defined for $t \in [0, 2]$.

(a) Let $F(t)$ be the Fourier series of $f(t)$ (meaning the 2-periodic extension of $f(t)$). Determine $F(2)$, $F\left(\frac{5}{2}\right)$ and $F\left(-\frac{1}{2}\right)$.

(b) Let $G(t)$ be the Fourier cosine series of $f(t)$. Determine $G(2)$, $G\left(\frac{5}{2}\right)$ and $G\left(-\frac{1}{2}\right)$.

(c) Let $H(t)$ be the Fourier sine series of $f(t)$. Determine $H(2)$, $H\left(\frac{5}{2}\right)$ and $H\left(-\frac{1}{2}\right)$.

Solution.

(a) Note that the extension of $f(t)$ has discontinuities at $\ldots, -2, 0, 2, 4, \ldots$ (see plot in previous example) and recall that the Fourier series takes average values at these discontinuities:
$F(2) = \frac{1}{2}(F(2^-) + F(2^+)) = \frac{1}{2}(0 + 4) = 2$
$F\left(\frac{5}{2}\right) = F\left(\frac{5}{2} - 2\right) = f\left(\frac{1}{2}\right) = \frac{15}{4}$
$F\left(-\frac{1}{2}\right) = F\left(-\frac{1}{2} + 2\right) = f\left(\frac{3}{2}\right) = \frac{7}{4}$

(b) $G(2) = f(2) = 0$ (see plot!)
[Note that $G(2^+) = G(2^+ - 4) = G(-2^+) = G(2^-)$ where we used that G is even in the last step; in fact, we can show like this that the Fourier cosine series of a continuous function is always continuous.]
$G\left(\frac{5}{2}\right) = G\left(\frac{5}{2} - 4\right) = G\left(-\frac{3}{2}\right) = f\left(\frac{3}{2}\right) = \frac{7}{4}$
$G\left(-\frac{1}{2}\right) = f\left(\frac{1}{2}\right) = \frac{15}{4}$

(c) $H(2) = \frac{1}{2}(f(2^-) - f(2^-)) = 0$ (see plot!)
[Note that $H(2^+) = H(2^+ - 4) = H(-2^+) = -H(2^-)$ where we used that H is odd in the last step; in fact, we can show like this that the Fourier sine series of a continuous function is always 0 at the jumps.]
$H\left(\frac{5}{2}\right) = H\left(\frac{5}{2} - 4\right) = H\left(-\frac{3}{2}\right) = -f\left(\frac{3}{2}\right) = -\frac{7}{4}$
$H\left(-\frac{1}{2}\right) = -f\left(\frac{1}{2}\right) = -\frac{15}{4}$

Fourier series and linear differential equations

In the following examples, we consider inhomogeneous linear DEs $p(D)y = F(t)$ where $F(t)$ is a periodic function that can be expressed as a Fourier series. We first review the notion of **resonance** (and how to predict it) and then solve such DEs.

Context. Recall that the inhomogeneous DE $my'' + ky = F(t)$ describes, for instance, the motion of a mass m on a spring with spring constant k under the influence of an external force $F(t)$. (Note how each term in the DE corresponds to a force: $my'' = ma$ from Newton's second law, ky from Hooke's law for springs, and $F(t)$ the external force.)

Example 143. Consider the linear DE $my'' + ky = \cos(\omega t)$. For which (external) **frequencies** $\omega > 0$ does **resonance** occur?

Solution. The characteristic roots (the roots of $p(D) = mD^2 + k$) are $\pm i\sqrt{k/m}$. Correspondingly, the solutions of the homogeneous equation $my'' + ky = 0$ are combinations of $\cos(\omega_0 t)$ and $\sin(\omega_0 t)$, where $\omega_0 = \sqrt{k/m}$ (ω_0 is called the **natural frequency** of the DE). Resonance occurs in the case $\omega = \omega_0$ when the external frequency matches the natural frequency.

Review. If $\omega \neq \omega_0$ (overlapping roots), then there is particular solution of the form $y_p(t) = A\cos(\omega t) + B\sin(\omega t)$ (for specific values of A and B). The general solution is $y(t) = A\cos(\omega t) + B\sin(\omega t) + C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)$, which is a bounded function of t. In contrast, if $\omega = \omega_0$, then the general solution is $y(t) = (C_1 + At)\cos(\omega_0 t) + (C_2 + Bt)\sin(\omega_0 t)$ and this function is unbounded.

Example 144. A mass-spring system is described by the DE $2y'' + 32y = \sum_{n=1}^{\infty} \frac{\cos(n\omega t)}{n^2 + 1}$.

For which ω does resonance occur?

Solution. The roots of $p(D) = 2D^2 + 32$ are $\pm 4i$, so that that the natural frequency is 4. Resonance therefore occurs if 4 equals $n\omega$ for some $n \in \{1, 2, 3, \ldots\}$. Equivalently, resonance occurs if $\omega = 4/n$ for some $n \in \{1, 2, 3, \ldots\}$.

Example 145. A mass-spring system is described by the DE $my'' + y = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left(\frac{nt}{3}\right)$.

For which m does resonance occur?

Solution. The roots of $p(D) = mD^2 + 1$ are $\pm i/\sqrt{m}$, so that the natural frequency is $1/\sqrt{m}$. Resonance therefore occurs if $1/\sqrt{m} = n/3$ for some $n \in \{1, 2, 3, \ldots\}$. Equivalently, resonance occurs if $m = 9/n^2$ for some $n \in \{1, 2, 3, \ldots\}$.

Though it requires some effort, we already know how to solve $p(D)y = F(t)$ for periodic forces $F(t)$, once we have a Fourier series for $F(t)$. The same approach works for linear equations of higher order, or even systems of equations.

Example 146. Find a particular solution of $2y'' + 32y = F(t)$, with $F(t) = \begin{cases} 10 & \text{if } t \in (0,1) \\ -10 & \text{if } t \in (1,2) \end{cases}$, extended 2-periodically.

Solution.

- From earlier, we already know $F(t) = 10\sum_{n \text{ odd}} \frac{4}{\pi n}\sin(\pi n t)$.
- We next solve the equation $2y'' + 32y = \sin(\pi n t)$ for $n = 1, 3, 5, \ldots$. First, we note that the external frequency is πn, which is never equal to the natural frequency $\omega_0 = 4$. Hence, there exists a particular solution of the form $y_p(t) = A\cos(\pi n t) + B\sin(\pi n t)$. To determine the coefficients A, B, we plug into the DE. Noting that $y_p'' = -\pi^2 n^2\, y_p$ (can you see why without computing two derivatives?), we get

  $$2y_p'' + 32y_p = (32 - 2\pi^2 n^2)(A\cos(\pi n t) + B\sin(\pi n t)) \stackrel{!}{=} \sin(\pi n t).$$

  We conclude $A = 0$ and $B = \frac{1}{32 - 2\pi^2 n^2}$, so that $y_p(t) = \frac{\sin(\pi n t)}{32 - 2\pi^2 n^2}$.
- We combine the particular solutions found in the previous step, to see that

  $$2y'' + 32y = 10\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n}\sin(\pi n t) \quad \text{is solved by} \quad y_p = 10\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n}\frac{\sin(\pi n t)}{32 - 2\pi^2 n^2}.$$

Example 147. Find a particular solution of $2y'' + 32y = F(t)$, with $F(t)$ the 2π-periodic function such that $F(t) = 10t$ for $t \in (-\pi, \pi)$.

Solution.

- The Fourier series of $F(t)$ is $F(t) = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{20}{n}\sin(nt)$. [Exercise!]
- We next solve the equation $2y'' + 32y = \sin(nt)$ for $n = 1, 2, 3, \ldots$. Note, however, that **resonance** occurs for $n = 4$, so we need to treat that case separately. If $n \neq 4$ then we find, as in the previous example, that $y_p(t) = \frac{\sin(nt)}{32 - 2n^2}$. [Note how this fails for $n = 4$!]

  For $2y'' + 32y = \sin(4t)$, we begin with $y_p = At\cos(4t) + Bt\sin(4t)$. Then $y_p' = (A + 4Bt)\cos(4t) + (B - 4At)\sin(4t)$, and $y_p'' = (8B - 16At)\cos(4t) + (-8A - 16Bt)\sin(4t)$. Plugging into the DE, we get $2y_p'' + 32y_p = 16B\cos(4t) - 16A\sin(4t) \stackrel{!}{=} \sin(4t)$, and thus $B = 0$, $A = -\frac{1}{16}$. So, $y_p = -\frac{1}{16}t\cos(4t)$.
- We combine the particular solutions to get that our DE

  $$2y'' + 32y = -5\sin(4t) + \sum_{\substack{n=1 \\ n \neq 4}}^{\infty} (-1)^{n+1}\frac{20}{n}\sin(nt)$$

  is solved by

  $$y_p(t) = \frac{5}{16}t\cos(4t) + \sum_{\substack{n=1 \\ n \neq 4}}^{\infty} (-1)^{n+1}\frac{20}{n}\frac{\sin(nt)}{32 - 2n^2}.$$

  As in the previous example, this solution cannot really be simplified. Make some plots to appreciate the dominating character of the term resulting from resonance!

Armin Straub
straub@southalabama.edu

Excursion: The Riemann hypothesis—A Millennium Prize Problem

The **Riemann zeta function** is defined by $\zeta(s) = \sum_{n \geqslant 1} \frac{1}{n^s}$.

Note that this series converges (for real s) if and only if $s > 1$.
The divergent series $\zeta(1)$ is the harmonic series, and $\zeta(p)$ is often called a p-series in Calculus II.

Example 148. Recall from Example 140 that using Fourier series, we were able to find that $\frac{\pi^2}{8} = \frac{1}{1} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$ Use this to derive $\zeta(2) = \sum_{n \geqslant 1} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Solution. If we split the sum into those terms where n is even and those where n is odd, then we get

$$\sum_{n \geqslant 1} \frac{1}{n^2} = \sum_{n \geqslant 1} \frac{1}{(2n)^2} + \sum_{n \geqslant 1} \frac{1}{(2n-1)^2} = \frac{1}{4} \sum_{n \geqslant 1} \frac{1}{n^2} + \frac{\pi^2}{8}.$$

If we write $x = \sum_{n \geqslant 1} \frac{1}{n^2}$, then this means that $x = \frac{1}{4}x + \frac{\pi^2}{8}$. We can then solve this equation to find $x = \frac{\pi^2}{6}$, which is what we wanted to derive.

Comment. Euler achieved worldwide fame in 1734 by discovering and proving that $\zeta(2) = \frac{\pi^2}{6}$ (as well as $\sum_{n \geqslant 1} \frac{1}{n^4} = \frac{\pi^4}{90}$ and similar formulas for $\zeta(6), \zeta(8), \ldots$).
Comment. On the other hand, no such evaluations are known for $\zeta(3), \zeta(5), \ldots$ and we don't even know (for sure) whether these are rational numbers. Nobody believes these to be rational numbers, but it was only in 1978 that Apéry proved that $\zeta(3)$ is not a rational number.

The Clay Mathematics Institute has offered 10^6 dollars each for the first correct solution to seven **Millennium Prize Problems**. Six of the seven problems remain open.

https://en.wikipedia.org/wiki/Millennium_Prize_Problems

Comment. Grigori Perelman solved the Poincaré conjecture in 2003 (but refused the prize money in 2010).
https://en.wikipedia.org/wiki/Poincaré_conjecture

The Riemann hypothesis is one of the seven Millennium Prize Problems and is concerned with the zeros of the Riemann zeta function $\zeta(s)$.

Recall that $\zeta(1)$ is the harmonic series, which diverges. For complex values of $s \neq 1$, there is a unique way to "analytically continue" the function $\zeta(s)$ from the definition $\zeta(s) = \sum_{n \geqslant 1} \frac{1}{n^s}$ which only works for $\operatorname{Re} s > 1$.

It is then "easy" to see that $\zeta(-2) = 0$, $\zeta(-4) = 0$, ... These are called the trivial zeros of $\zeta(s)$.

The **Riemann hypothesis** claims that all other zeros of $\zeta(s)$ lie on the (vertical) line $s = \frac{1}{2} + ai$ (where a is real).
A proof of this conjecture (checked for the first 10,000,000,000 zeroes) is worth $1,000,000.
http://www.claymath.org/millennium-problems/riemann-hypothesis

The reason for caring about the zeros is that they are intimately tied to the distribution of primes. The prime number theorem states that, up to x, there are about $x / \ln(x)$ many primes. The Riemann hypothesis gives very precise error estimates for an improved prime number theorem (using a function more complicated than the logarithm).

The connection to primes. Here's a vague indication that $\zeta(s)$ is intimately connected to prime numbers:

$$\begin{aligned}\zeta(s)=\sum_{n\geqslant 1}\frac{1}{n^s} &= \left(1+\frac{1}{2^s}+\frac{1}{2^{2s}}+\ldots\right)\left(1+\frac{1}{3^s}+\frac{1}{3^{2s}}+\ldots\right)\left(1+\frac{1}{5^s}+\frac{1}{5^{2s}}+\ldots\right)\cdots\\ &= \frac{1}{1-2^{-s}}\frac{1}{1-3^{-s}}\frac{1}{1-5^{-s}}\cdots\\ &= \prod_{p\text{ prime}}\frac{1}{1-p^{-s}}\end{aligned}$$

To see that the second equality holds, imagine multiplying out the right-hand side. For instance, we get the term $\frac{1}{2^s}\cdot\frac{1}{3^{4s}}\cdot\frac{1}{7^s}=\frac{1}{(2\cdot 3^4\cdot 7)^s}$. This matches the term $\frac{1}{n^s}$ on the left-hand side for $n=2\cdot 3^4\cdot 7$. Since every positive integer n has a unique factorization into prime factors, this matching is one-to-one.

The final infinite product is called the Euler product for the zeta function.

If the Riemann hypothesis was true, then we would be better able to estimate the number $\pi(x)$ of primes $p\leqslant x$. More generally, certain statements about the zeta function can be translated to statements about primes. For instance, the (non-obvious!) fact that $\zeta(s)$ has no zeros for $\operatorname{Re} s=1$ implies the prime number theorem.

http://www-users.math.umn.edu/~garrett/m/v/pnt.pdf

Boundary value problems

Example 149. The **IVP** (initial value problem) $y'' + 4y = 0$, $y(0) = 0$, $y'(0) = 0$ has the unique solution $y(x) = 0$.

Initial value problems are often used when the problem depends on time. Then, $y(0)$ and $y'(0)$ describe the initial configuration at $t = 0$.

For problems which instead depend on spatial variables, such as position, it may be natural to specify values at positions on the boundary (for instance, if $y(x)$ describes the steady-state temperature of a rod at position x, we might know the temperature at the two end points).

The next example illustrates that such a boundary value problem (BVP) may or may not have a unique solution.

Example 150. Verify the following claims.

(a) The BVP $y'' + 4y = 0$, $y(0) = 0$, $y(1) = 0$ has the unique solution $y(x) = 0$.

(b) The BVP $y'' + \pi^2 y = 0$, $y(0) = 0$, $y(1) = 0$ is solved by $y(x) = B\sin(\pi x)$ for any value B.

Solution.

(a) We know that the general solution to the DE is $y(x) = A\cos(2x) + B\sin(2x)$. The boundary conditions imply $y(0) = A \overset{!}{=} 0$ and, using that $A = 0$, $y(1) = B\sin(2) \overset{!}{=} 0$ shows that $B = 0$ as well.

(b) This time, the general solution to the DE is $y(x) = A\cos(\pi x) + B\sin(\pi x)$. The boundary conditions imply $y(0) = A \overset{!}{=} 0$ and, using that $A = 0$, $y(1) = B\sin(\pi) \overset{!}{=} 0$. This second condition is true for every B.

It is therefore natural to ask: for which λ does the BVP $y'' + \lambda y = 0$, $y(0) = 0$, $y(L) = 0$ have nonzero solutions? (We assume that $L > 0$.)

Such solutions are called **eigenfunctions** and λ is the corresponding **eigenvalue**.

Remark. Compare that to our previous use of the term eigenvalue: given a matrix A, we asked: for which λ does $A\boldsymbol{v} - \lambda\boldsymbol{v} = 0$ have nonzero solutions $\boldsymbol{v}$? Such solutions were called eigenvectors and λ was the corresponding eigenvalue.

Example 151. Find all eigenfunctions and eigenvalues of $y'' + \lambda y = 0$, $y(0) = 0$, $y(L) = 0$.

Such a problem is called an **eigenvalue problem**.

Solution. The solutions of the DE look different in the cases $\lambda < 0$, $\lambda = 0$, $\lambda > 0$, so we consider them individually.

$\boldsymbol{\lambda = 0}$. Then $y(x) = Ax + B$ and $y(0) = y(L) = 0$ implies that $y(x) = 0$. No eigenfunction here.

$\boldsymbol{\lambda < 0}$. The roots of the characteristic polynomial are $\pm\sqrt{-\lambda}$. Writing $\rho = \sqrt{-\lambda}$, the general solution therefore is $y(x) = Ae^{\rho x} + Be^{-\rho x}$. $y(0) = A + B \overset{!}{=} 0$ implies $B = -A$. Using that, we get $y(L) = A(e^{\rho L} - e^{-\rho L}) \overset{!}{=} 0$. For eigenfunctions we need $A \neq 0$, so $e^{\rho L} = e^{-\rho L}$ which implies $\rho L = -\rho L$. This cannot happen since $\rho \neq 0$ and $L \neq 0$. Again, no eigenfunctions in this case.

$\boldsymbol{\lambda > 0}$. The roots of the characteristic polynomial are $\pm i\sqrt{\lambda}$. Writing $\rho = \sqrt{\lambda}$, the general solution thus is $y(x) = A\cos(\rho x) + B\sin(\rho x)$. $y(0) = A \overset{!}{=} 0$. Using that, $y(L) = B\sin(\rho L) \overset{!}{=} 0$. Since $B \neq 0$ for eigenfunctions, we need $\sin(\rho L) = 0$. This happens if $\rho L = n\pi$ for $n = 1, 2, 3, \ldots$ (since ρ and L are both positive, n must be positive as well). Equivalently, $\rho = \frac{n\pi}{L}$. Consequently, we find the eigenfunctions $y_n(x) = \sin\frac{n\pi x}{L}$, $n = 1, 2, 3, \ldots$, with eigenvalue $\lambda = \left(\frac{n\pi}{L}\right)^2$.

Armin Straub
straub@southalabama.edu

Example 152. Suppose that a rod of length L is compressed by a force P (with ends being pinned [not clamped] down). We model the shape of the rod by a function $y(x)$ on some interval $[0, L]$. The theory of elasticity predicts that, under certain simplifying assumptions, y should satisfy $EIy'' + Py = 0$, $y(0) = 0$, $y(L) = 0$.

Here, EI is a constant modeling the inflexibility of the rod (E, known as Young's modulus, depends on the material, and I depends on the shape of cross-sections (it is the area moment of inertia)).

In other words, $y'' + \lambda y = 0$, $y(0) = 0$, $y(L) = 0$, with $\lambda = \frac{P}{EI}$.

The fact that there is no nonzero solution unless $\lambda = (\frac{\pi n}{L})^2$ for some $n = 1, 2, 3, \ldots$, means that buckling can only occur if $P = (\frac{\pi n}{L})^2 EI$. In particular, no buckling occurs for forces less than $\frac{\pi^2 EI}{L^2}$. This is known as the critical load (or Euler load) of the rod.

Comment. This is a very simplified model. In particular, it assumes that the deflections are small. (Technically, the buckled rod in our model is longer than L; of course, that's not the case in practice.)

https://en.wikipedia.org/wiki/Euler%27s_critical_load

Example 153. Find all eigenfunctions and eigenvalues of

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(3) = 0.$$

Solution. We distinguish three cases:

- $\boldsymbol{\lambda < 0}$. The characteristic roots are $\pm r = \pm\sqrt{-\lambda}$ and the general solution to the DE is $y(x) = Ae^{rx} + Be^{-rx}$. Then $y'(0) = Ar - Br = 0$ implies $B = A$, so that $y(3) = A(e^{3r} + e^{-3r})$. Since $e^{3r} + e^{-3r} > 0$, we see that $y(3) = 0$ only if $A = 0$. So there is no solution for $\lambda < 0$.
- $\boldsymbol{\lambda = 0}$. The general solution to the DE is $y(x) = A + Bx$. Then $y'(0) = 0$ implies $B = 0$, and it follows from $y(3) = A = 0$ that $\lambda = 0$ is not an eigenvalue.
- $\boldsymbol{\lambda > 0}$. The characteristic roots are $\pm i\sqrt{\lambda}$. So, with $r = \sqrt{\lambda}$, the general solution is $y(x) = A\cos(rx) + B\sin(rx)$. $y'(0) = Br = 0$ implies $B = 0$. Then $y(3) = A\cos(3r) = 0$. Note that $\cos(3r) = 0$ is true if and only if $3r = \frac{\pi}{2} + n\pi = \frac{(2n+1)\pi}{2}$ for some integer n. Since $r > 0$, we have $n \geqslant 0$. Correspondingly, $\lambda = r^2 = \left(\frac{(2n+1)\pi}{6}\right)^2$ and $y(x) = \cos\left(\frac{(2n+1)\pi}{6}x\right)$.

In summary, we have that the eigenvalues are $\lambda = \left(\frac{(2n+1)\pi}{6}\right)^2$, with $n = 0, 1, 2, \ldots$ with corresponding eigenfunctions $y(x) = \cos\left(\frac{(2n+1)\pi}{6}x\right)$.

Partial differential equations

The heat equation

We wish to describe one-dimensional heat flow.

Comment. If this sounds very specialized, it might help to know that the heat equation is also used, for instance, in probability (Brownian motion), financial math (Black-Scholes), or chemical processes (diffusion equation).

Let $u(x, t)$ describe the temperature at time t at position x.

If we model a heated rod of length L, then $x \in [0, L]$.

Notation. $u(x, t)$ depends on two variables. When taking derivatives, we will use the notations $u_t = \frac{\partial}{\partial t}u$ and $u_{xx} = \frac{\partial^2}{\partial x^2}u$ for first and higher derivatives.

Experience tells us that heat flows from warmer to cooler areas and has an averaging effect.

Make a sketch of some temperature profile $u(x,t)$ for fixed t.

As t increases, we expect maxima (where $u_{xx} < 0$) of that profile to flatten out (which means that $u_t < 0$); similarly, minima (where $u_{xx} > 0$) should go up (meaning that $u_t > 0$). The simplest relationship between u_t and u_{xx} which conforms with our expectation is $u_t = k u_{xx}$, with $k > 0$.

(heat equation)

$$u_t = k u_{xx}$$

Note that the heat equation is a linear and homogeneous **partial differential equation**.

In particular, the principle of superposition holds: if u_1 and u_2 solve the heat equation, then so does $c_1 u_1 + c_2 u_2$.

Higher dimensions. In higher dimensions, the heat equation takes the form $u_t = k(u_{xx} + u_{yy})$ or $u_t = k(u_{xx} + u_{yy} + u_{zz})$. The heat equation is often written as $u_t = k\Delta u$, where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplace operator you may know from Calculus III.

The Laplacian Δu is also often written as $\Delta u = \nabla^2 u$. The operator $\nabla = (\partial / \partial x, \partial / \partial y)$ is pronounced "nabla" (Greek for a certain harp) or "del" (Persian for heart), and ∇^2 is short for the inner product $\nabla \cdot \nabla$.

Example 154. Note that $u(x,t) = ax + b$ solves the heat equation.

Example 155. To get a feeling, let us find some other solutions to $u_t = u_{xx}$ (for starters, $k = 1$).

- For instance, $u(x,t) = e^t e^x$ is a solution.
  [Not a very interesting one for modeling heat flow because it increases exponentially in time.]
- ... to be continued ...
  Can you find further solutions?

Let us think about what is needed to describe a unique solution of the heat equation.

- **Initial condition** at $t = 0$: $u(x,0) = f(x)$ (IC)

  This specifies an initial temperature distribution at time $t = 0$.

- **Boundary condition** at $x = 0$ and $x = L$: (BC)

  Assuming that heat only enters/exits at the boundary (think of our rod as being insulated, except possibly at the two ends), we need some condition on the temperature at the ends. For instance:

  - $u(0,t) = A$, $u(L,t) = B$

    This models a rod where one end is kept at temperature A and the other end at temperature B.

  - $u_x(0,t) = u_x(L,t) = 0$

    This models a rod whose ends are insulated as well.

Under such assumptions, our physical intuition suggests that there should be a unique solution.

Important comment. We can always transform the case $u(0,t) = A$, $u(L,t) = B$ into $u(0,t) = u(L,t) = 0$ by using the fact that $u(t,x) = ax + b$ solves $u_t = k u_{xx}$. Can you spell this out?

Review. The heat equation: $u_t = k u_{xx}$

Example 156. (cont'd) To get a feeling, let us find some solutions to $u_t = u_{xx}$.

- $u(x,t) = ax + b$ is a solution.
- For instance, $u(x,t) = e^t e^x$ is a solution.
  [Not a very interesting one for modeling heat flow because it increases exponentially in time.]
- More interesting are $u(x,t) = e^{-t}\cos(x)$ and $u(x,t) = e^{-t}\sin(x)$.
- More generally, $e^{-n^2 t}\cos(nx)$ and $e^{-n^2 t}\sin(nx)$ are solutions.

Important observation. This actually reveals a strategy for solving the PDE $u_t = u_{xx}$ with conditions such as:

$$\begin{array}{ll} u(0,t) = u(\pi,t) = 0 & \text{(BC)} \\ u(x,0) = f(x), \quad x \in (0,\pi) & \text{(IC)} \end{array}$$

Namely, the solutions $u_n(x,t) = e^{-n^2 t}\sin(nx)$ all satisfy (BC).
It remains to satisfy (IC). Note that $u_n(x,0) = \sin(nx)$. To find $u(x,t)$ such that $u(x,0) = f(x)$, we can write $f(x)$ as a Fourier sine series (i.e. extend $f(x)$ to a 2π-periodic odd function):

$$f(x) = \sum_{n \geqslant 1} b_n \sin(nx)$$

Then $u(x,t) = \sum_{n \geqslant 1} b_n u_n(x,t) = \sum_{n \geqslant 1} b_n e^{-n^2 t}\sin(nx)$ solves the PDE $u_t = u_{xx}$ with (BC) and (IC).

Example 157. Find the unique solution $u(x,t)$ to: $\begin{array}{lr} u_t = u_{xx} & \text{(PDE)} \\ u(0,t) = u(\pi,t) = 0 & \text{(BC)} \\ u(x,0) = \sin(2x) - 7\sin(3x), \quad x \in (0,\pi) & \text{(IC)} \end{array}$

Solution. As we just observed (see next example for what to do in general), the functions $u_n(x,t) = e^{-n^2 t}\sin(nx)$ satisfy (PDE) and (BC) for any integer $n = 1, 2, 3, \ldots$

Since $u_n(x,0) = \sin(nx)$, we have $u_2(x,0) - 7u_3(x,0) = \sin(2x) - 7\sin(3x)$ as needed for (IC).

Therefore, (PDE)+(BC)+(IC) is solved by

$$u(x,t) = u_2(x,t) - 7u_3(x,t) = e^{-4t}\sin(2x) - 7e^{-9t}\sin(3x).$$

Comment. Why did we restrict the integer n to the case $n \geqslant 1$?
[$n = 0$ just gives the zero function, and negative values don't give anything new because $u_{-n}(x,t) = -u_n(x,t)$.]

In the next example, we show that this idea can always be used to solve the heat equation.

Armin Straub
straub@southalabama.edu

Example 158. Find the unique solution $u(x,t)$ to:
$$\begin{aligned} u_t &= k u_{xx} && \text{(PDE)} \\ u(0,t) &= u(L,t) = 0 && \text{(BC)} \\ u(x,0) &= f(x), \quad x \in (0,L) && \text{(IC)} \end{aligned}$$

Solution.

- We will first look for simple solutions of (PDE)+(BC) (and then we plan to take a combination of such solutions that satisfies (IC) as well). Namely, we look for solutions $u(x,t) = X(x)T(t)$. This approach is called **separation of variables** and it is crucial for solving other PDEs as well.

- Plugging into (PDE), we get $X(x)T'(t) = kX''(x)T(t)$, and so $\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)}$.

  Note that the two sides cannot depend on x (because the right-hand side doesn't) and they cannot depend on t (because the left-hand side doesn't). Hence, they have to be constant. Let's call this constant $-\lambda$.

  Then, $\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)} = \text{const} =: -\lambda$.

  We thus have $X'' + \lambda X = 0$ and $T' + \lambda kT = 0$.

- Consider (BC). Note that $u(0,t) = X(0)T(t) = 0$ implies $X(0) = 0$.

  [Because otherwise $T(t) = 0$ for all t, which would mean that $u(x,t)$ is the dull zero solution.]

  Likewise, $u(L,t) = X(L)T(t) = 0$ implies $X(L) = 0$.

- So X solves $X'' + \lambda X = 0$, $X(0) = 0$, $X(L) = 0$. We know that, up to multiples, the only nonzero solutions are the eigenfunctions $X(x) = \sin\left(\frac{\pi n}{L}x\right)$ corresponding to the eigenvalues $\lambda = \left(\frac{\pi n}{L}\right)^2$, $n = 1, 2, 3, \ldots$.

- On the other hand, T solves $T' + \lambda kT = 0$, and hence $T(t) = e^{-\lambda kt} = e^{-\left(\frac{\pi n}{L}\right)^2 kt}$.

- Taken together, we have the solutions $u_n(x,t) = e^{-\left(\frac{\pi n}{L}\right)^2 kt} \sin\left(\frac{\pi n}{L}x\right)$ solving (PDE)+(BC).

- We wish to combine these in such a way that (IC) holds as well.

  At $t = 0$, $u_n(x,0) = \sin\left(\frac{\pi n}{L}x\right)$. All of these are $2L$-periodic.

  Hence, we extend $f(x)$, which is only given on $(0, L)$, to an odd $2L$-periodic function (its Fourier sine series!). By making it odd, its Fourier series will only involve sine terms: $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n}{L}x\right)$. Note that

  $$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x,$$

  where the first integral makes reference to the extension of $f(x)$ while the second integral only uses $f(x)$ on its original interval of definition.

Consequently, (PDE)+(BC)+(IC) is solved by

$$u(x,t) = \sum_{n=1}^{\infty} b_n\, u_n(x,t) = \sum_{n=1}^{\infty} b_n\, e^{-\left(\frac{\pi n}{L}\right)^2 kt} \sin\left(\frac{\pi n}{L}x\right),$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x.$$

Example 159. Find the unique solution $u(x,t)$ to:
$$\begin{aligned} & u_t = 3u_{xx} && \text{(PDE)} \\ & u(0,t) = u(4,t) = 0 && \text{(BC)} \\ & u(x,0) = 5\sin(\pi x) - \sin(3\pi x), \quad x \in (0,4) && \text{(IC)} \end{aligned}$$

Solution. This is the case $k=3$, $L=4$ that we solved in Example 158 where we found that the functions

$$u_n(x,t) = e^{-\left(\frac{\pi n}{L}\right)^2 kt} \sin\!\left(\frac{\pi n}{L} x\right) = e^{-3\left(\frac{\pi n}{4}\right)^2 t} \sin\!\left(\frac{\pi n}{4} x\right)$$

solve (PDE)+(BC). Since $u_n(x,0) = \sin\!\left(\frac{\pi n}{4} x\right)$, we have

$$5u_4(x,0) - u_{12}(x,0) = 5\sin(\pi x) - \sin(3\pi x),$$

which is what we need for the right-hand side of (IC). Therefore, (PDE)+(BC)+(IC) is solved by

$$u(x,t) = 5u_4(x,t) - u_{12}(x,t) = 5e^{-3\pi^2 t} \sin(\pi x) - e^{-27\pi^2 t} \sin(3\pi x).$$

Example 160. Find the unique solution $u(x,t)$ to: $\begin{array}{l} u_t = u_{xx} \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = 1, \quad x \in (0,1) \end{array}$

Solution. This is the case $k = 1$, $L = 1$ and $f(x) = 1$, $x \in (0,1)$, of Example 158.

In the final step, we extend $f(x)$ to the 2-periodic odd function of Example 137. In particular, earlier, we have already computed that the Fourier series is

$$f(x) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \sin(n\pi x).$$

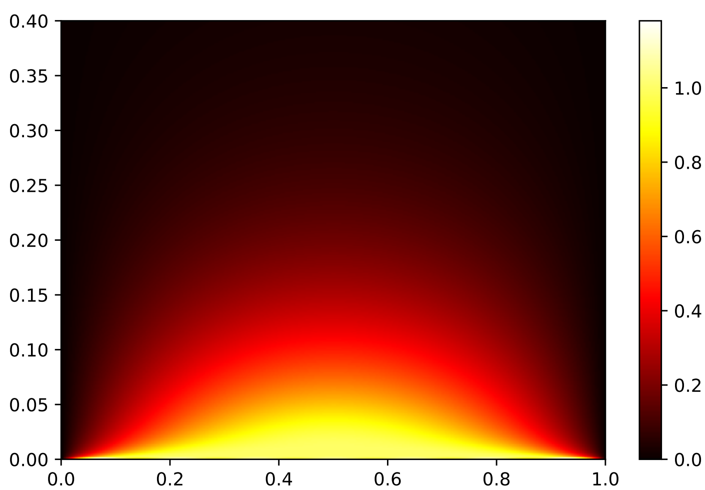
Hence, $u(x,t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} e^{-\pi^2 n^2 t} \sin(n\pi x)$.

Comment. Note that, for $t > 0$, the exponential very quickly approaches 0 (because of the $-n^2$ in the exponent), so that we get very accurate approximations with only a handful terms.

We can use Sage to plot our solution using the terms $n = 1, 3, 5, \ldots, 19$ of the infinite sum:

```
>>> var('x,t');
>>> uxt = sum(4/(pi*n) * exp(-pi^2*n^2*t) * sin(pi*n*x) for n in range(1,20,2))
>>> density_plot(uxt, (x,0,1), (t,0,0.4), plot_points=200, cmap='hot')
```

The resulting plot should look similar to the following:



Can you make sense of the plot? Does that plot confirm our expectations?

[Note that the horizontal axis shows x for $x \in (0,1)$, while the vertical axis shows t for $t \in (0,0.4)$. Yellow represents 1 (for $t = 0$, all values are 1 because of the initial condition), while black represents 0.]

The boundary conditions in the next example model insulated ends.

Example 161. Find the unique solution $u(x,t)$ to:
$$\begin{aligned} &u_t = k u_{xx} && \text{(PDE)}\\ &u_x(0,t) = u_x(L,t) = 0 && \text{(BC)}\\ &u(x,0) = f(x), \quad x \in (0,L) && \text{(IC)} \end{aligned}$$

Solution.

- We proceed as before and look for solutions $u(x,t) = X(x)T(t)$ (**separation of variables**).

  Plugging into (PDE), we get $X(x)T'(t) = kX''(x)T(t)$, and so $\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)} = \text{const} =: -\lambda$.
  We thus have $X'' + \lambda X = 0$ and $T' + \lambda k T = 0$.

- From the (BC), i.e. $u_x(0,t) = X'(0)T(t) = 0$, we get $X'(0) = 0$.
  Likewise, $u_x(L,t) = X'(L)T(t) = 0$ implies $X'(L) = 0$.

- So X solves $X'' + \lambda X = 0$, $X'(0) = 0$, $X'(L) = 0$. It is left as a homework to show that, up to multiples, the only nonzero solutions of this eigenvalue problem are $X(x) = \cos(\frac{\pi n}{L} x)$ corresponding to $\lambda = (\frac{\pi n}{L})^2$, $n = 0, 1, 2, 3 \ldots$. [A similar problem with full solution will be on the practice problems.]

- On the other hand (as before), T solves $T' + \lambda k T = 0$, and hence $T(t) = e^{-\lambda k t} = e^{-(\frac{\pi n}{L})^2 k t}$.

- Taken together, we have the solutions $u_n(x,t) = e^{-(\frac{\pi n}{L})^2 k t} \cos(\frac{\pi n}{L} x)$ solving (PDE)+(BC).

- We wish to combine these in such a way that (IC) holwhereds.
  At $t = 0$, $u_n(x,0) = \cos(\frac{\pi n}{L} x)$. All of these are $2L$-periodic.

  Hence, we extend $f(x)$, which is only given on $(0,L)$, to an even $2L$-periodic function (its Fourier cosine series!). By making it even, its Fourier series only involves cosine terms: $f(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos(\frac{\pi n}{L} x)$.
  Note that
  $$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x,$$
  where the first integral makes reference to the extension of $f(x)$ while the second integral only uses $f(x)$ on its original interval of definition.

Consequently, (PDE)+(BC)+(IC) is solved by
$$u(x,t) = \frac{a_0}{2} u_0(x,t) + \sum_{n=1}^{\infty} a_n u_n(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-(\frac{\pi n}{L})^2 k t} \cos\left(\frac{\pi n}{L} x\right),$$
where
$$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x.$$

Example 162. Find the unique solution $u(x,t)$ to:
$$\begin{aligned} &u_t = 3u_{xx} && \text{(PDE)}\\ &u_x(0,t) = u_x(4,t) = 0 && \text{(BC)}\\ &u(x,0) = 2 + 5\cos(\pi x) - \cos(3\pi x), \ x \in (0,4) && \text{(IC)} \end{aligned}$$

Solution. This is the case $k = 3$, $L = 4$ that we solved in Example 161 where we found that the functions
$$u_n(x,t) = e^{-(\frac{\pi n}{L})^2 k t} \cos\left(\frac{\pi n}{L} x\right) = e^{-3(\frac{\pi n}{4})^2 t} \cos\left(\frac{\pi n}{4} x\right)$$
solve (PDE)+(BC). Since $u_n(x,0) = \cos(\frac{\pi n}{4} x)$, we have
$$2u_0(x,0) + 5u_4(x,0) - u_{12}(x,0) = 2 + 5\cos(\pi x) - \cos(3\pi x),$$
which is what we need for the right-hand side of (IC). Therefore, (PDE)+(BC)+(IC) is solved by
$$u(x,t) = 2u_0(x,t) + 5u_4(x,t) - u_{12}(x,t) = 2 + 5e^{-3\pi^2 t} \cos(\pi x) - e^{-27\pi^2 t} \cos(3\pi x).$$

The inhomogeneous heat equation

We next indicate that we can similarly solve the inhomogeneous heat equation (with inhomogeneous boundary conditions).

Comment. We indicated earlier that

$$\begin{array}{lr} u_t = k u_{xx} & \text{(PDE)} \\ u(0,t) = a, \quad u(L,t) = b & \text{(BC)} \\ u(x,0) = f(x), \quad x \in (0,L) & \text{(IC)} \end{array}$$

can be solved by realizing that $Ax + B$ solves (PDE).

Indeed, let $v(x) = a + \frac{b-a}{L}x$ (so that $v(0) = a$ and $v(L) = b$). We then look for a solution of the form $u(x,t) = v(x) + w(x,t)$. Note that $u(x,t)$ solves (PDE)+(BC)+(IC) if and only if $w(x,t)$ solves:

$$\begin{array}{lr} w_t = k w_{xx} & \text{(PDE)} \\ w(0,t) = 0, \quad w(L,t) = 0 & \text{(BC*)} \\ w(x,0) = f(x) - v(x), \quad x \in (0,L) & \text{(IC)} \end{array}$$

This is the (homogeneous) heat equation that we know how to solve.

$v(x)$ is called the **steady-state solution** (it does not depend on time!) and $w(x,t)$ the **transient solution** (note that $w(x,t)$ and its partial derivatives tend to zero as $t \to \infty$ because of the boundary conditions (BC*)).

Example 163. Consider the heat flow problem: $\begin{array}{lr} u_t = 3u_{xx} + 4x^2 & \text{(PDE)} \\ u(0,t) = 1, \quad u_x(3,t) = -5 & \text{(BC)} \\ u(x,0) = f(x), \quad x \in (0,3) & \text{(IC)} \end{array}$

Determine the steady-state solution and spell out equations characterizing the transient solution.

Solution. We look for a solution of the form $u(x,t) = v(x) + w(x,t)$, where $v(x)$ is the steady-state solution and where $w(x,t)$ is the transient solution which (together with its derivatives) tends to zero as $t \to \infty$.

- Plugging into (PDE), we get $w_t = 3v'' + 3w_{xx} + 4x^2$. Letting $t \to \infty$, this becomes $0 = 3v'' + 4x^2$. Note that this also implies that $w_t = 3w_{xx}$.
- Plugging into (BC), we get $v(0) + w(0,t) = 1$ and $v'(3) + w_x(3,t) = -5$. Letting $t \to \infty$, these become $v(0) = 1$ and $v'(3) = -5$.
- Solving the ODE $0 = 3v'' + 4x^2$, we find

  $$v(x) = \iint -\frac{4}{3}x^2 \mathrm{d}x\mathrm{d}x = \int \left(-\frac{4}{9}x^3 + C\right)\mathrm{d}x = -\frac{1}{9}x^4 + Cx + D.$$

  The boundary conditions $v(0) = 1$ and $v'(3) = -5$ imply $D = 1$ and $-\frac{4}{9} \cdot 3^3 + C = -5$ (so that $C = 7$). In conclusion, the steady-state solution is $v(x) = -\frac{1}{9}x^4 + 1 + 7x$.

On the other hand, the transient solution $w(x,t)$ is characterized as the unique solution to:

$$\begin{array}{lr} w_t = 3w_{xx} & \text{(PDE*)} \\ w(0,t) = 0, \quad w_x(3,t) = 0 & \text{(BC*)} \\ w(x,0) = f(x) - v(x) & \text{(IC*)} \end{array}$$

This homogeneous heat flow problem can now be solved using separation of variables.

Example 164. For $t \geqslant 0$ and $x \in [0, 4]$, consider the heat flow problem: $\begin{aligned} u_t &= 2u_{xx} + e^{-x/2} \\ u_x(0,t) &= 3 \\ u(4,t) &= -2 \\ u(x,0) &= f(x) \end{aligned}$

Determine the steady-state solution and spell out equations characterizing the transient solution.

Solution. We look for a solution of the form $u(x,t) = v(x) + w(x,t)$, where $v(x)$ is the steady-state solution and where the transient solution $w(x,t)$ tends to zero as $t \to \infty$ (as do its derivatives).

- Plugging into (PDE), we get $w_t = 2v'' + 2w_{xx} + e^{-x/2}$. Letting $t \to \infty$, this becomes $0 = 2v'' + e^{-x/2}$.
- Plugging into (BC), we get $w_x(0,t) + v'(0) = 3$ and $w(4,t) + v(4) = -2$.
  Letting $t \to \infty$, these become $v'(0) = 3$ and $v(4) = -2$.
- Solving the ODE $0 = 2v'' + e^{-x/2}$, we find

  $$v(x) = \iint -\frac{1}{2}e^{-x/2}\mathrm{d}x\mathrm{d}x = \int (e^{-x/2} + C)\mathrm{d}x = -2e^{-x/2} + Cx + D.$$

  The boundary conditions $v'(0) = 3$ and $v(4) = -2$ imply $C = 2$ and $-2e^{-2} + 8 + D = -2$.
  In conclusion, the steady-state solution is $v(x) = -2e^{-x/2} + 2x - 10 + 2e^{-2}$.

On the other hand, the transient solution $w(x,t)$ is characterized as the unique solution to:

$$\begin{aligned} &w_t = 2w_{xx} \\ &w_x(0,t) = 0, \quad w(4,t) = 0 \\ &w(x,0) = f(x) - v(x) \end{aligned}$$

Note. We know how to solve this homogeneous heat equation using separation of variables.

Armin Straub
straub@southalabama.edu

Steady-state temperature: The Laplace equation

(2D and 3D heat equation) In higher dimensions, the heat equation takes the form $u_t = k(u_{xx} + u_{yy})$ or $u_t = k(u_{xx} + u_{yy} + u_{zz})$.

The heat equation is often written as $u_t = k\Delta u$ where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ (2D) or $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ (3D) is the **Laplace operator** you may know from Calculus III.

Other notations. $\Delta u = \operatorname{div} \operatorname{grad} u = \nabla \cdot \nabla u = \nabla^2 u$

If temperature is steady, then $u_t = 0$. Hence, the steady-state temperature $u(x, y)$ must satisfy the PDE $u_{xx} + u_{yy} = 0$.

(Laplace equation, 2D)

$$u_{xx} + u_{yy} = 0$$

Comment. The Laplace equation is so important that its solutions have their own name: **harmonic functions**.

It is also known as the "potential equation"; satisfied by electric/gravitational potential functions. (More generally, such potentials, if not in the vacuum, satisfy the **Poisson equation** $u_{xx} + u_{yy} = f(x, y)$, the inhomogeneous version of the Laplace equation.)

Recall from Calculus III (if you have taken that class) that the gradient of a scalar function $f(x, y)$ is the vector field $\boldsymbol{F} = \operatorname{grad} f = \nabla f = \begin{bmatrix} f_x(x, y) \\ f_y(x, y) \end{bmatrix}$. One says that $\boldsymbol{F}$ is a **gradient field** and f is a **potential function** for $\boldsymbol{F}$ (for instance, $\boldsymbol{F}$ could be a gravitational field with gravitational potential f).

The divergence of a vector field $\boldsymbol{G} = \begin{bmatrix} g(x, y) \\ h(x, y) \end{bmatrix}$ is $\operatorname{div} \boldsymbol{G} = g_x + h_y$. One also writes $\operatorname{div} \boldsymbol{G} = \nabla \cdot \boldsymbol{G}$.

The gradient field of a scalar function f is divergence-free if and only if f satisfies the Laplace equation $\Delta f = 0$.

One way to describe a unique solution to the Laplace equation within a region is by specifying the values of $u(x, y)$ along the boundary of that region.

This is particularly natural for steady-state temperatures profiles of a region R. The Laplace equation governs how temperature behaves inside the region but we need to also prescribe the temperature on the boundary.

The PDE with such a boundary condition is called a Dirichlet problem:

(Dirichlet problem)

$u_{xx} + u_{yy} = 0$ within region R

$u(x, y) = f(x, y)$ on boundary of R

In general. A Dirichlet problem consists of a PDE, that needs to hold within a region R, and prescribed values on the boundary of that region ("Dirichlet boundary conditions").

Finite difference method: A glance at discretizing PDEs

We know from Calculus that $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

PDEs quickly become impossibly difficult to approach with exact solution techniques.

It is common therefore to proceed numerically. One approach is to discretize the problem.

For instance. We could use $f'(x) \approx \frac{1}{h}[f(x+h) - f(x)]$ to replace $f'(x)$ with the **finite difference** on the RHS.

Such approximate methods are called **finite difference methods**.

Finite difference methods are a common approach to numerically solving PDEs.
The ODE or PDE translates into a (sparse) system of linear equations which is then solved using Linear Algebra.

Example 165.

- $f'(x) \approx \frac{1}{h}[f(x+h) - f(x)]$ is a **forward difference** for $f'(x)$.
- $f'(x) \approx \frac{1}{h}[f(x) - f(x-h)]$ is a **backward difference** for $f'(x)$.
- $f'(x) \approx \frac{1}{2h}[f(x+h) - f(x-h)]$ is a **central difference** for $f'(x)$.
  Note that this is the average of the forward and the backward difference. The calculations below show that the central difference performs better as an approximation of $f'(x)$.

Comment. Recall that power series $f(x)$ have the Taylor expansion $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$.

Equivalently, $f(x+h) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} h^n = f(x) + h f'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{6} f'''(x) + O(h^4)$. It follows that

$$\frac{1}{h}[f(x+h) - f(x)] = f'(x) + \boxed{\frac{h}{2} f''(x) + O(h^2)} = f'(x) + \boxed{O(h)}.$$

The $\boxed{\text{error}}$ is of order $O(h)$. On the other hand, combining

$$\begin{aligned} f(x+h) &= f(x) + h f'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{6} f'''(x) + O(h^4), \\ f(x-h) &= f(x) - h f'(x) + \frac{h^2}{2} f''(x) - \frac{h^3}{6} f'''(x) + O(h^4), \end{aligned}$$

it follows that

$$\frac{1}{2h}[f(x+h) - f(x-h)] = f'(x) + \boxed{\frac{h^2}{6} f'''(x) + O(h^3)} = f'(x) + \boxed{O(h^2)}.$$

The $\boxed{\text{error}}$ is of order $O(h^2)$.

Comment. An error of order h^2 means that if we cut h by a factor of, say, $\frac{1}{10}$, then we expect the error to be cut by a factor of $\frac{1}{10^2} = \frac{1}{100}$.

Armin Straub
straub@southalabama.edu

Example 166. Find a central difference for $f''(x)$.

Solution. Adding the two expansions for $f(x+h)$ and $f(x-h)$ to kill the $f'(x)$ terms, and subtracting $2f(x)$, we find that

$$\frac{1}{h^2}[f(x+h)-2f(x)+f(x-h)] = f''(x) + \boxed{\frac{h^2}{12}f^{(4)}(x)+O(h^3)} = f''(x) + \boxed{O(h^2)}.$$

The $\boxed{\text{error}}$ is of order 2.

Alternatively. If we iterate the approximation $f'(x) \approx \frac{1}{2h}[f(x+h)-f(x-h)]$ (in the second step, we apply it with x replaced by $x \pm h$), we obtain

$$f''(x) \approx \frac{1}{2h}[f'(x+h)-f'(x-h)] \approx \frac{1}{4h^2}[f(x+2h)-2f(x)+f(x-2h)],$$

which is the same as what we found above, just with h replaced by $2h$.

Example 167. (discretizing Δ) Use the above central difference approximation for second derivatives to derive a finite difference for $\Delta u = u_{xx}+u_{yy}$ in 2D.

Solution.

$$\begin{aligned} u_{xx} + u_{yy} &\approx \frac{1}{h^2}[u(x+h,y)-2u(x,y)+u(x-h,y)] + \frac{1}{h^2}[u(x,y+h)-2u(x,y)+u(x,y-h)] \\ &= \frac{1}{h^2}[u(x+h,y)+u(x-h,y)+u(x,y+h)+u(x,y-h)-4u(x,y)] \end{aligned}$$

Notation. This finite difference is typically represented as $\frac{1}{h^2}\begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix}$, the **five-point stencil**.

Comment. Recall that solutions to $\Delta u = 0$ are supposed to describe steady-state temperature distributions. We can see from our discretization that this is reasonable. Namely, $\Delta u = 0$ becomes approximately equivalent to

$$u(x,y) = \frac{1}{4}(u(x+h,y)+u(x-h,y)+u(x,y+h)+u(x,y-h)).$$

In other words, the temperature $u(x,y)$ at a point (x,y) should be the average of the temperatures of its four "neighbors" $u(x+h,y)$ (right), $u(x-h,y)$ (left), $u(x,y+h)$ (top), $u(x,y-h)$ (bottom).

Comment. Think about how to use this finite difference to numerically solve the corresponding Dirichlet problem by discretizing (one equation per lattice point).

Advanced comment. If $\Delta u = 0$ then, when discretizing, the center point has the average value of the four points adjacent to it. This leads to the **maximum principle**: if $\Delta u = 0$ on a region R, then the maximum (and, likewise, minimum) value of u must occur at a boundary point of R.

Armin Straub
straub@southalabama.edu

Example 168. Discretize the following Dirichlet problem:

$$\begin{aligned} u_{xx} + u_{yy} &= 0 \quad \text{(PDE)} \\ u(x,0) &= 2 \\ u(x,2) &= 3 \qquad \text{(BC)} \\ u(0,y) &= 0 \\ u(1,y) &= 0 \end{aligned}$$

Use a step size of $h = \frac{1}{3}$.

Comment. Note that, for the Dirichlet problem as well as for our discretization, it doesn't matter that the boundary conditions aren't well-defined at the corners.

Solution. Note that our rectangle has side lengths 1 (in x direction) and 2 (in y direction). With a step size of $h = \frac{1}{3}$ we therefore get $4 \cdot 7$ lattice points, namely the points

$$u_{m,n} = u(mh, nh), \quad m \in \{0,1,2,3\}, \; n \in \{0,1,\ldots,6\}.$$

Further note that the boundary conditions determine the values of $u_{m,n}$ if $m = 0$ or $m = 3$ as well as if $n = 0$ or $n = 6$. This leaves $2 \cdot 5 = 10$ points at which we need to determine the value of $u_{m,n}$.

Next, we approximate $u_{xx} + u_{yy}$ by $\frac{1}{h^2}[u(x+h,y) + u(x-h,y) + u(x,y+h) + u(x,y-h) - 4u(x,y)]$ (see previous example for how we obtained this finite difference approximation). Note that, if $u(x,y) = u_{m,n}$ is one of our lattice points, then the other four terms in the finite difference are lattice points as well; for instance, $u(x+h,y) = u_{m+1,n}$. The equation $u_{xx} + u_{yy} = 0$ therefore translates into

$$u_{m+1,n} + u_{m-1,n} + u_{m,n+1} + u_{m,n-1} - 4u_{m,n} = 0.$$

Spelling out these equation for each $m \in \{1,2\}$ and $n \in \{1,2,\ldots,5\}$, we get 10 (linear) equations for our 10 unknown values. For instance, here are the equations for $(m,n) = (1,1)$, $(1,2)$ as well as $(2,5)$:

$$\begin{aligned} u_{2,1} + \underbrace{u_{0,1}}_{=0} + u_{1,2} + \underbrace{u_{1,0}}_{=2} - 4u_{1,1} &= 0 \\ u_{2,2} + \underbrace{u_{0,2}}_{=0} + u_{1,3} + u_{1,1} - 4u_{1,2} &= 0 \\ &\vdots \\ \underbrace{u_{3,5}}_{=0} + u_{1,5} + \underbrace{u_{2,6}}_{=3} + u_{2,4} - 4u_{2,5} &= 0 \end{aligned}$$

In matrix-vector form, these linear equations take the form:

$$\begin{bmatrix} -4 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ & & & & \vdots & & & & & \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} u_{1,1} \\ u_{1,2} \\ u_{1,3} \\ u_{1,4} \\ u_{1,5} \\ u_{2,1} \\ u_{2,2} \\ u_{2,3} \\ u_{2,4} \\ u_{2,5} \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ \vdots \\ -3 \end{bmatrix}$$

Solving this system, we find $u_{1,1} \approx 0.7847$, $u_{1,2} \approx 0.3542$, ..., $u_{2,5} \approx 1.1597$.

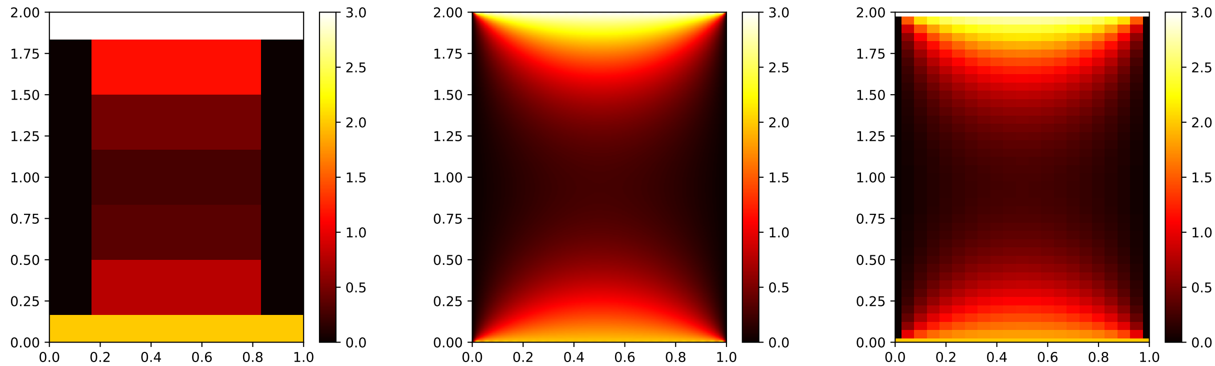
For comparison, the corresponding exact values are $u\left(\frac{1}{3}, \frac{1}{3}\right) \approx 0.7872$, $u\left(\frac{1}{3}, \frac{2}{3}\right) \approx 0.3209$, ..., $u\left(\frac{2}{3}, \frac{5}{3}\right) \approx 1.1679$. These were computed from the exact formula

$$u(x,y) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \frac{\sin(\pi n x)}{1 - e^{4\pi n}} [2(e^{\pi n y} - e^{-\pi n (y-4)}) + 3(e^{\pi n (2-y)} - e^{\pi n (2+y)})],$$

which we will derive soon.

Armin Straub
straub@southalabama.edu

The three plots below visualize the discretized solution with $h=\frac{1}{3}$ from Example 168, the exact solution, as well as the discretized solution with $h=\frac{1}{20}$.



Comment. The first plot looks a bit overly rough because we chose not to interpolate the values. As we showed above, the approximate values at the ten lattice points are actually pretty decent for such a large step size.

Warning. The resulting linear systems quickly become very large. For instance, if we use a step size of $h=\frac{1}{100}$, then we need to determine roughly $100\cdot 200=20{,}000$ ($99\cdot 199$ to be exact) values $u_{m,n}$. The corresponding matrix M will have about $20{,}000^2=400{,}000{,}000$ entries, which is already challenging for a weak machine if we use generic linear algebra software. At this point it is important to realize that most entries of the matrix M are 0. Such matrices are called **sparse** and there are efficient algorithms for solving systems involving such matrices.

Example 169. Discretize the following Dirichlet problem:

$$\begin{aligned} u_{xx}+u_{yy} &= 0 \quad \text{(PDE)} \\ u(x,0) &= 2 \\ u(x,1) &= 3 \\ u(0,y) &= 1 \qquad \text{(BC)} \\ u(2,y) &= 4 \end{aligned}$$

Use a step size of $h=\frac{1}{2}$.

Solution. Note that our rectangle has side lengths 2 (in x direction) and 1 (in y direction). With a step size of $h=\frac{1}{2}$ we therefore get $5\cdot 3$ lattice points, namely the points

$$u_{m,n}=u(mh,nh), \quad m\in\{0,1,2,3,4\},\ n\in\{0,1,2\}.$$

Further note that the boundary conditions determine the values of $u_{m,n}$ if $m=0$ or $m=4$ as well as if $n=0$ or $n=2$. This leaves $3\cdot 1=3$ points at which we need to determine the value of $u_{m,n}$.

If we approximate $u_{xx}+u_{yy}$ by $\frac{1}{h^2}[u(x+h,y)+u(x-h,y)+u(x,y+h)+u(x,y-h)-4u(x,y)]$ then, in terms of our lattice points, the equation $u_{xx}+u_{yy}=0$ translates into

$$u_{m+1,n}+u_{m-1,n}+u_{m,n+1}+u_{m,n-1}-4u_{m,n}=0.$$

Spelling out these equation for each $m\in\{1,2,3\}$ and $n=1$, we get 3 equations for our 3 unknown values:

$$\begin{aligned} u_{2,1}+\underbrace{u_{0,1}}_{=1}+\underbrace{u_{1,2}}_{=3}+\underbrace{u_{1,0}}_{=2}-4u_{1,1} &= 0 \\ u_{3,1}+u_{1,1}+\underbrace{u_{2,2}}_{=3}+\underbrace{u_{2,0}}_{=2}-4u_{2,1} &= 0 \\ \underbrace{u_{4,1}}_{=4}+u_{2,1}+\underbrace{u_{3,2}}_{=3}+\underbrace{u_{3,0}}_{=2}-4u_{3,1} &= 0 \end{aligned}$$

In matrix-vector form, these linear equations take the form:

$$\begin{bmatrix} -4 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & -4 \end{bmatrix}\begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \end{bmatrix}=\begin{bmatrix} -6 \\ -5 \\ -9 \end{bmatrix}$$

Example 170. Consider the polygonal region with vertices $(0,0)$, $(4,0)$, $(4,2)$, $(2,2)$, $(2,3)$, $(0,3)$. We wish to find the steady-state temperature distribution $u(x,y)$ within this region if the temperature is A between $(0,0)$, $(4,0)$, and B elsewhere on the boundary.

Spell out the resulting equations when we discretize this problem using a step size of $h=1$.

Solution. As before, we write $u_{m,n}=u(mh,nh)$. Make a sketch!

$$\begin{array}{ccccc} & B & & & \\ B & u_{1,2} & B & B & \\ B & u_{1,1} & u_{2,1} & u_{3,1} & B \\ & A & A & A & \end{array}$$

If we approximate $u_{xx}+u_{yy}$ by $\frac{1}{h^2}[u(x+h,y)+u(x-h,y)+u(x,y+h)+u(x,y-h)-4u(x,y)]$ then, in terms of our lattice points, the equation $u_{xx}+u_{yy}=0$ translates into

$$u_{m+1,n}+u_{m-1,n}+u_{m,n+1}+u_{m,n-1}-4u_{m,n}=0.$$

Spelling out these equation in matrix-vector form, we obtain:

$$\begin{bmatrix} -4 & 1 & 0 & 1 \\ 1 & -4 & 1 & 0 \\ 0 & 1 & -4 & 0 \\ 1 & 0 & 0 & -4 \end{bmatrix}\begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \\ u_{1,2} \end{bmatrix} = \begin{bmatrix} -A-B \\ -A-B \\ -A-2B \\ -3B \end{bmatrix}$$

Comment. Note that, because of the way we discretize, it matters that there is a well-defined temperature at the boundary vertex $(2,2)$. For the other vertices, we don't need a well-defined temperature (and so it is not a problem that it is unclear what the temperature should be at $(0,0)$ or $(4,0)$ where it jumps from A to B).

Solving the Laplace equation inside a rectangle

One complication in solving the 2D Laplace equation inside a region R with Dirichlet boundary conditions is that the boundary of R is some curve (opposed to just two points in the 1D case).

A strategy to deal with complicated regions is to break them into simpler regions, such as rectangles or triangles.

Next, we demonstrate that we can fully solve the 2D Laplace equation in the case when R is a rectangle. In that case, the Dirichlet problem takes the form:

$$\begin{array}{rcll} u_{xx}+u_{yy} & = & 0 & \text{(PDE)} \\ u(x,0) & = & f_1(x) & \\ u(x,b) & = & f_2(x) & \text{(BC)} \\ u(0,y) & = & f_3(y) & \\ u(a,y) & = & f_4(y) & \end{array}$$

The first crucial observation is that we can break this problem into four parts:

$$\begin{array}{llll} u_{xx}+u_{yy}=0 & u_{xx}+u_{yy}=0 & u_{xx}+u_{yy}=0 & u_{xx}+u_{yy}=0 \\ u(x,0)=f_1(x) & u(x,0)=0 & u(x,0)=0 & u(x,0)=0 \\ u(x,b)=0 & u(x,b)=f_2(x) & u(x,b)=0 & u(x,b)=0 \\ u(0,y)=0 & u(0,y)=0 & u(0,y)=f_3(y) & u(0,y)=0 \\ u(a,y)=0 & u(a,y)=0 & u(a,y)=0 & u(a,y)=f_4(y) \end{array}$$

If we solve these four simpler Dirichlet problems, then the sum of the four solutions will solve the original Dirichlet problem.

Armin Straub
straub@southalabama.edu

As a consequence, it is no loss of generality to use homogeneous boundary conditions for three of the four sides. We illustrate how to solve this case in the next example. Our main tool is separation of variables, just as for the heat equation $u_t = k u_{xx}$.

Example 171. Find the unique solution $u(x,y)$ to:
$$\begin{array}{ll} u_{xx} + u_{yy} = 0 & \text{(PDE)} \\ u(x,0) = f(x) & \\ u(x,b) = 0 & \text{(BC)} \\ u(0,y) = 0 & \\ u(a,y) = 0 & \end{array}$$

Solution.

- We proceed as before and look for solutions $u(x,y) = X(x)Y(y)$ (**separation of variables**).

  Plugging into (PDE), we get $X''(x)Y(y) + X(x)Y''(y)$, and so $\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \text{const} =: -\lambda$.

  We thus have $X'' + \lambda X = 0$ and $Y'' - \lambda Y = 0$.

- From the last three (BC), we get $X(0) = 0$, $X(a) = 0$, $Y(b) = 0$.

  We ignore the first (inhomogeneous) condition for now.

- So X solves $X'' + \lambda X = 0$, $X(0) = 0$, $X(a) = 0$.

  From earlier, we know that, up to multiples, the only nonzero solutions of this eigenvalue problem are $X(x) = \sin(\frac{\pi n}{a} x)$ corresponding to $\lambda = (\frac{\pi n}{a})^2$, $n = 1, 2, 3 \ldots$.

- On the other hand, Y solves $Y'' - \lambda Y = 0$, and hence $Y(y) = A e^{\sqrt{\lambda} y} + B e^{-\sqrt{\lambda} y}$.

  The condition $Y(b) = 0$ implies that $A e^{\sqrt{\lambda} b} + B e^{-\sqrt{\lambda} b} = 0$ so that $B = -A e^{2\sqrt{\lambda} b}$.

  Hence, $Y(y) = A(e^{\sqrt{\lambda} y} - e^{\sqrt{\lambda}(2b - y)})$.

- Taken together, we have the solutions $u_n(x,y) = \sin(\frac{\pi n}{a} x)\left(e^{\frac{\pi n}{a} y} - e^{\frac{\pi n}{a}(2b - y)}\right)$ solving (PDE)+(BC), with the exception of $u(x,0) = f(x)$.

- We wish to combine these in such a way that $u(x,0) = f(x)$ holds as well.

  At $y = 0$, $u_n(x,0) = \sin(\frac{\pi n}{a} x)(1 - e^{2\pi n b/a})$. All of these are $2a$-periodic.

  Hence, we extend $f(x)$, which is only given on $(0,a)$, to an odd $2a$-periodic function (its Fourier sine series!). By making it odd, its Fourier series will only involve sine terms: $f(x) = \sum_{n=1}^{\infty} b_n \sin(\frac{\pi n}{a} x)$.

  Note that
  $$b_n = \frac{1}{a}\int_{-a}^{a} f(x)\sin\left(\frac{n\pi x}{a}\right)\mathrm{d}x = \frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)\mathrm{d}x,$$
  where the first integral makes reference to the extension of $f(x)$ while the second integral only uses $f(x)$ on its original interval of definition.

Consequently, (PDE)+(BC) is solved by
$$u(x,y) = \sum_{n=1}^{\infty} \frac{b_n}{1 - e^{2\pi n b/a}} u_n(x,y) = \sum_{n=1}^{\infty} \frac{b_n}{1 - e^{2\pi n b/a}} \sin\left(\frac{\pi n}{a} x\right)\left(e^{\frac{\pi n}{a} y} - e^{\frac{\pi n}{a}(2b - y)}\right),$$
where
$$b_n = \frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)\mathrm{d}x.$$

Example 172. Find the unique solution $u(x,y)$ to:

$$\begin{aligned} u_{xx}+u_{yy} &= 0 \quad \text{(PDE)} \\ u(x,0) &= 1 \\ u(x,2) &= 0 \\ u(0,y) &= 0 \quad\quad \text{(BC)} \\ u(1,y) &= 0 \end{aligned}$$

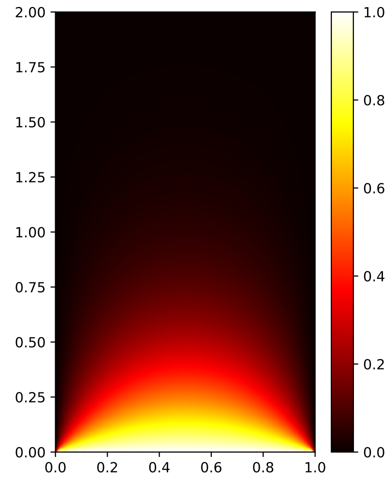
Solution. This is the special case of the previous example with $a=1$, $b=2$ and $f(x)=1$ for $x\in(0,1)$.

From Example 137, we know that $f(x)$ has the Fourier sine series

$$f(x)=\sum_{\substack{n=1\\ n\text{ odd}}}^{\infty}\frac{4}{\pi n}\sin(n\pi x),\qquad x\in(0,1).$$

Hence,

$$u(x,y)=\sum_{\substack{n=1\\ n\text{ odd}}}^{\infty}\frac{4}{\pi n}\frac{1}{1-e^{4\pi n}}\sin(\pi n x)(e^{\pi n y}-e^{\pi n(4-y)}).$$



Comment. The temperature at the center is $u(\frac{1}{2},1)\approx 0.0549$ (only the first term of the infinite sum suffices for this estimate; the first three terms suffice for 9 digits of accuracy).

Example 173. Find the unique solution $u(x,y)$ to:

$$\begin{aligned} u_{xx}+u_{yy} &= 0 \quad \text{(PDE)} \\ u(x,0) &= 0 \\ u(x,2) &= 3 \\ u(0,y) &= 0 \quad\quad \text{(BC)} \\ u(1,y) &= 0 \end{aligned}$$

Solution. Instead of starting from scratch (homework exercise!), let us reuse our computations:
Let $v(x,y)=u(x,2-y)$. Then $v_{xx}+v_{yy}=0$, $v(x,0)=3$, $v(x,2)=0$, $v(0,y)=0$, $v(1,y)=0$.
Hence, it follows from the previous example that

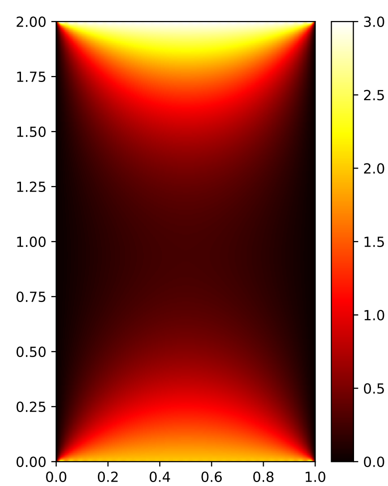
$$v(x,y)=3\sum_{\substack{n=1\\ n\text{ odd}}}^{\infty}\frac{4}{\pi n}\frac{1}{1-e^{4\pi n}}\sin(\pi n x)(e^{\pi n y}-e^{\pi n(4-y)}).$$

Consequently,

$$u(x,y)=v(x,2-y)=3\sum_{\substack{n=1\\ n\text{ odd}}}^{\infty}\frac{4}{\pi n}\frac{1}{1-e^{4\pi n}}\sin(\pi n x)(e^{\pi n(2-y)}-e^{\pi n(2+y)}).$$

Example 174. Find the unique solution $u(x,y)$ to:

$$\begin{aligned} u_{xx}+u_{yy} &= 0 \\ u(x,0)=2,\quad u(x,2) &= 3 \\ u(0,y)=0,\quad u(1,y) &= 0 \end{aligned}$$



Solution. Note that $u(x,y)$ is a combination of the solutions to the previous two examples!

$$u(x,y)=\sum_{\substack{n=1\\ n\text{ odd}}}^{\infty}\frac{4}{\pi n}\frac{\sin(\pi n x)}{1-e^{4\pi n}}[2(e^{\pi n y}-e^{-\pi n(y-4)})+3(e^{\pi n(2-y)}-e^{\pi n(2+y)})].$$

Example 175. Find the unique solution $u(x,y)$ to:

$$\begin{aligned} u_{xx}+u_{yy} &= 0 && \text{(PDE)}\\ u(x,0) &= 4\sin(\pi x)-5\sin(3\pi x) \\ u(x,2) &= 0 && \text{(BC)}\\ u(0,y) &= 0 \\ u(3,y) &= 0 \end{aligned}$$

Solution.

- We look for solutions $u(x,y)=X(x)Y(y)$ (**separation of variables**).
  Plugging into (PDE), we get $X''(x)Y(y)+X(x)Y''(y)$, and so $\frac{X''(x)}{X(x)}=-\frac{Y''(y)}{Y(y)}=\text{const}$.
  We thus have $X''-\text{const}\,X=0$ and $Y''+\text{const}\,Y=0$.
- From the last three (BC), we get $X(0)=0$, $X(3)=0$, $Y(2)=0$.
- So X solves $X''+\lambda X=0$ (we choose $\lambda=-\text{const}$), $X(0)=0$, $X(3)=0$.
  From earlier (or do it!), we know that, up to multiples, the only nonzero solutions of this eigenvalue problem are $X(x)=\sin\left(\frac{1}{3}\pi n x\right)$ corresponding to $\lambda=\left(\frac{1}{3}\pi n\right)^2$, $n=1,2,3\ldots$.
- On the other hand, Y solves $Y''-\lambda Y=0$, and hence $Y(y)=Ae^{\sqrt{\lambda}y}+Be^{-\sqrt{\lambda}y}$.
  The condition $Y(2)=0$ implies that $Ae^{2\sqrt{\lambda}}+Be^{-2\sqrt{\lambda}}=0$ so that $B=-Ae^{4\sqrt{\lambda}}$.
  Hence, $Y(y)=A(e^{\sqrt{\lambda}y}-e^{\sqrt{\lambda}(4-y)})=A\left(e^{\frac{1}{3}\pi n y}-e^{\frac{1}{3}\pi n(4-y)}\right)$.
- Taken together, we have the solutions $u_n(x,y)=\sin\left(\frac{1}{3}\pi n x\right)\left(e^{\frac{1}{3}\pi n y}-e^{\frac{1}{3}\pi n(4-y)}\right)$ solving (PDE)+(BC), with the exception of $u(x,0)=4\sin(\pi x)-5\sin(3\pi x)$.
- At $y=0$, $u_n(x,0)=\sin\left(\frac{1}{3}\pi n x\right)\left(1-e^{\frac{4}{3}\pi n}\right)$.
  In particular, $u_3(x,0)=\sin(\pi x)(1-e^{4\pi})$ and $u_9(x,0)=\sin(3\pi x)(1-e^{12\pi})$.
  Hence, $4\sin(\pi x)-5\sin(3\pi x)=\frac{4}{1-e^{4\pi}}u_3(x,0)-\frac{5}{1-e^{12\pi}}u_9(x,0)$. Therefore our overall solution is

$$\begin{aligned} u(x,y) &= \frac{4}{1-e^{4\pi}}u_3(x,y)-\frac{5}{1-e^{12\pi}}u_9(x,y)\\ &= \frac{4}{1-e^{4\pi}}\sin(\pi x)(e^{\pi y}-e^{\pi(4-y)})-\frac{5}{1-e^{12\pi}}\sin(3\pi x)(e^{3\pi y}-e^{3\pi(4-y)}). \end{aligned}$$

Comment. Of course, in general, our inhomogeneous (BC) will be a function $f(x)$ that is not such an obvious combination of our special solutions $u_n(x,0)$. In that case, we need to compute an appropriate Fourier expansion of $f(x)$ first (here, the Fourier sine series of $f(x)$).

Hyperbolic sine and cosine

Review. Euler's formula states that $e^{it} = \cos(t) + i\sin(t)$.

Recall that a function $f(t)$ is **even** if $f(-t) = f(t)$. Likewise, it is **odd** if $f(-t) = -t$.

Note that $f(t) = t^n$ is even if and only if n is even. Likewise, $f(t) = t^n$ is odd if and only if n is odd. That's where the names are coming from.

Any function $f(t)$ can be decomposed into an even and an odd part as follows:

$$f(t) = f_{\text{even}}(t) + f_{\text{odd}}(t), \quad f_{\text{even}}(t) = \frac{1}{2}(f(t) + f(-t)), \quad f_{\text{odd}}(t) = \frac{1}{2}(f(t) - f(-t)).$$

Verify that $f_{\text{even}}(t)$ indeed is even, and that $f_{\text{odd}}(t)$ indeed is an odd function (regardless of $f(t)$).

Example 176. The **hyperbolic cosine**, denoted $\cosh(t)$, is the even part of e^t. Likewise, the **hyperbolic sine**, denoted $\sinh(t)$, is the odd part of e^t.

- Equivalently, $\cosh(t) = \frac{1}{2}(e^t + e^{-t})$ and $\sinh(t) = \frac{1}{2}(e^t - e^{-t})$.
- In particular, $e^t = \cosh(t) + \sinh(t)$.

  As recalled above, any function is the sum of its even and odd part.

  Comparing with Euler's formula, we find $\cosh(it) = \cos(t)$ and $\sinh(it) = i\sin(t)$. This indicates that $\cosh$ and $\sinh$ are related to $\cos$ and $\sin$, as their name suggests (see below for the "hyperbolic" part).
- $\frac{\mathrm{d}}{\mathrm{d}t}\cosh(t) = \sinh(t)$ and $\frac{\mathrm{d}}{\mathrm{d}t}\sinh(t) = \cosh(t)$.
- $\cosh(t)$ and $\sinh(t)$ both satisfy the DE $y'' = y$.

  We can write the general solution as $C_1 e^t + C_2 e^{-t}$ or, if useful, as $C_1\cosh(t) + C_2\sinh(t)$.
- $\cosh(t)^2 - \sinh(t)^2 = 1$

  Verify this by substituting $\cosh(t) = \frac{1}{2}(e^t + e^{-t})$ and $\sinh(t) = \frac{1}{2}(e^t - e^{-t})$.

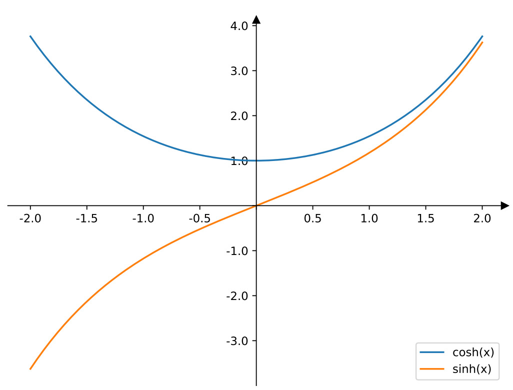
  Note that the equation $x^2 - y^2 = 1$ describes a **hyperbola** (just like $x^2 + y^2 = 1$ describes a circle).

  The above equation is saying that $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cosh(t) \\ \sinh(t) \end{bmatrix}$ is a parametrization of the hyperbola.

  Comment. Circles and hyperbolas are conic sections (as are ellipses and parabolas).

  Comment. Hyperbolic geometry plays an important role, for instance, in special relativity:
  https://en.wikipedia.org/wiki/Hyperbolic_geometry

Homework. Write down the parallel properties of cosine and sine.



Armin Straub
straub@southalabama.edu

The fin equation from thermodynamics

The following is an example from thermodynamics. The governing differential equation is a second-order DE that is like the equation describing the motion of a mass on a spring ($my''+ky=0$) except that one term has the opposite sign. Besides showcasing an application, we want to show off how $\cosh$ and $\sinh$ are useful for writing certain solutions in a more pleasing form.

Let $T(x)$ describe the temperature at position x in a fin with fin base at $x=0$ and fin tip at $x=L$.

For more context on fins: https://en.wikipedia.org/wiki/Fin_(extended_surface)

If we write $\theta(x)=T(x)-T_\infty$ for the temperature excess at position x (with T_∞ the external temperature), then we find (under various simplifying assumptions) that the temperature distribution in our fin satisfies the following DE, known as the **fin equation**:

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2}-m^2\theta=0, \quad m^2=\frac{hP}{kA}>0.$$

- A is the cross-sectional area of the fin (assumed to be the same for all positions x).
- P is the perimeter of the fin (assumed to be the same for all positions x).
- k is the thermal conductivity of the material (assumed to be constant).
- h is the convection heat transfer coefficient (assumed to be constant).

Since the DE is homogeneous and linear with characteristic roots $\pm m$, the general solution is

$$\theta(x)=C_1e^{mx}+C_2e^{-mx}=D_1\cosh(mx)+D_2\sinh(mx).$$

The constants C_1, C_2 (or, equivalently, D_1, D_2) can then be found by emposing appropriate boundary conditions at the **fin base** ($x=0$) and at the **fin tip** ($x=L$).

In practice, we often know the temperature at the fin base and therefore the temperature excess, resulting in the boundary condition $\theta(0)=\theta_0$. At the fin tip, common boundary conditions are:

- $\theta(L)\to 0$ as $L\to\infty$ (infinitely long fin)

  In this case, the fin is so long that the temperature at the fin tip approaches the external temperature. Mathematically, we get $\theta(x)=Ce^{-mx}$ since $e^{mx}\to\infty$ as $x\to\infty$. It follows from $\theta(0)=\theta_0$ that $C=\theta_0$.

  Thus, the temperature excess is $\theta(x)=\theta_0\, e^{-mx}$.

- $\theta'(L)=0$ (neglible heat loss at the fin tip, "adiabatic fin tip")

  This can be a more reasonable assumption than the infinitely long fin. Note that the total heat transfer from the fin is proportional to its surface area. If the surface area at the fin tip is a negligible fraction of the total surface area, then it is reasonable to assume that $\theta'(L)=0$.

  In this case, the temperature excess is $\theta(x)=\theta_0\frac{\cosh(m(L-x))}{\cosh(mL)}$.

  Check! Instead of computing this from scratch (do that as well, later!), check that this indeed solves the DE as well as the boundary conditions $\theta(0)=\theta_0$ and $\theta'(L)=0$. This should be a rather quick check!

- $\theta(L)=\theta_L$ (specified temperature at fin tip)

  In this case, the temperature excess is $\theta(x)=\frac{\theta_L\sinh(mx)+\theta_0\sinh(m(L-x))}{\sinh(mL)}$.

  Check! Again, check that this indeed solves the DE as well as the boundary conditions $\theta(0)=\theta_0$ and $\theta(L)=\theta_L$. Once more, this should be a quick and pleasant check.