## Hyperbolic sine and cosine

**Review.** Euler's formula states that $e^{it} = \cos(t) + i\sin(t)$.

Recall that a function $f(t)$ is **even** if $f(-t) = f(t)$. Likewise, it is **odd** if $f(-t) = -t$.

Note that $f(t) = t^n$ is even if and only if $n$ is even. Likewise, $f(t) = t^n$ is odd if and only if $n$ is odd. That's where the names are coming from.

Any function $f(t)$ can be decomposed into an even and an odd part as follows:

$$f(t) = f_{\text{even}}(t) + f_{\text{odd}}(t), \quad f_{\text{even}}(t) = \frac{1}{2}(f(t) + f(-t)), \quad f_{\text{odd}}(t) = \frac{1}{2}(f(t) - f(-t)).$$

Verify that $f_{\text{even}}(t)$ indeed is even, and that $f_{\text{odd}}(t)$ indeed is an odd function (regardless of $f(t)$).

**Example 176.** The **hyperbolic cosine**, denoted $\cosh(t)$, is the even part of $e^t$. Likewise, the **hyperbolic sine**, denoted $\sinh(t)$, is the odd part of $e^t$.

- Equivalently, $\cosh(t) = \frac{1}{2}(e^t + e^{-t})$ and $\sinh(t) = \frac{1}{2}(e^t - e^{-t})$.
- In particular, $e^t = \cosh(t) + \sinh(t)$.

  As recalled above, any function is the sum of its even and odd part.

  Comparing with Euler's formula, we find $\cosh(it) = \cos(t)$ and $\sinh(it) = i\sin(t)$. This indicates that $\cosh$ and $\sinh$ are related to $\cos$ and $\sin$, as their name suggests (see below for the "hyperbolic" part).
- $\frac{\mathrm{d}}{\mathrm{d}t}\cosh(t) = \sinh(t)$ and $\frac{\mathrm{d}}{\mathrm{d}t}\sinh(t) = \cosh(t)$.
- $\cosh(t)$ and $\sinh(t)$ both satisfy the DE $y'' = y$.

  We can write the general solution as $C_1 e^t + C_2 e^{-t}$ or, if useful, as $C_1\cosh(t) + C_2\sinh(t)$.
- $\cosh(t)^2 - \sinh(t)^2 = 1$

  Verify this by substituting $\cosh(t) = \frac{1}{2}(e^t + e^{-t})$ and $\sinh(t) = \frac{1}{2}(e^t - e^{-t})$.

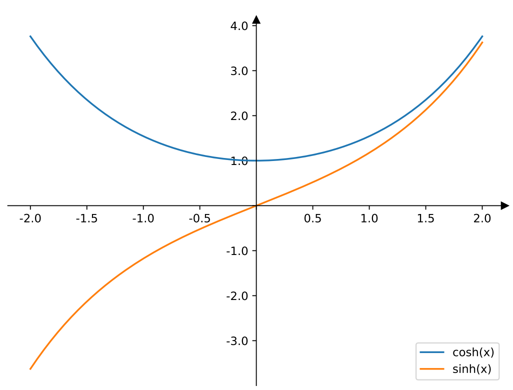
  Note that the equation $x^2 - y^2 = 1$ describes a **hyperbola** (just like $x^2 + y^2 = 1$ describes a circle).

  The above equation is saying that $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cosh(t) \\ \sinh(t) \end{bmatrix}$ is a parametrization of the hyperbola.

  **Comment.** Circles and hyperbolas are conic sections (as are ellipses and parabolas).

  **Comment.** Hyperbolic geometry plays an important role, for instance, in special relativity:
  https://en.wikipedia.org/wiki/Hyperbolic_geometry

**Homework.** Write down the parallel properties of cosine and sine.



## The fin equation from thermodynamics

The following is an example from thermodynamics. The governing differential equation is a second-order DE that is like the equation describing the motion of a mass on a spring ($my'' + ky = 0$) except that one term has the opposite sign. Besides showcasing an application, we want to show off how $\cosh$ and $\sinh$ are useful for writing certain solutions in a more pleasing form.

Let $T(x)$ describe the temperature at position $x$ in a fin with fin base at $x = 0$ and fin tip at $x = L$.

For more context on fins: https://en.wikipedia.org/wiki/Fin_(extended_surface)

If we write $\theta(x) = T(x) - T_\infty$ for the temperature excess at position $x$ (with $T_\infty$ the external temperature), then we find (under various simplifying assumptions) that the temperature distribution in our fin satisfies the following DE, known as the **fin equation**:

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} - m^2\theta = 0, \quad m^2 = \frac{hP}{kA} > 0.$$

- $A$ is the cross-sectional area of the fin (assumed to be the same for all positions $x$).
- $P$ is the perimeter of the fin (assumed to be the same for all positions $x$).
- $k$ is the thermal conductivity of the material (assumed to be constant).
- $h$ is the convection heat transfer coefficient (assumed to be constant).

Since the DE is homogeneous and linear with characteristic roots $\pm m$, the general solution is

$$\theta(x) = C_1 e^{mx} + C_2 e^{-mx} = D_1 \cosh(mx) + D_2 \sinh(mx).$$

The constants $C_1, C_2$ (or, equivalently, $D_1, D_2$) can then be found by emposing appropriate boundary conditions at the **fin base** ($x = 0$) and at the **fin tip** ($x = L$).

In practice, we often know the temperature at the fin base and therefore the temperature excess, resulting in the boundary condition $\theta(0) = \theta_0$. At the fin tip, common boundary conditions are:

- $\theta(L) \to 0$ as $L \to \infty$ (infinitely long fin)

  In this case, the fin is so long that the temperature at the fin tip approaches the external temperature. Mathematically, we get $\theta(x) = Ce^{-mx}$ since $e^{mx} \to \infty$ as $x \to \infty$. It follows from $\theta(0) = \theta_0$ that $C = \theta_0$.

  Thus, the temperature excess is $\theta(x) = \theta_0 e^{-mx}$.

- $\theta'(L) = 0$ (neglible heat loss at the fin tip, "adiabatic fin tip")

  This can be a more reasonable assumption than the infinitely long fin. Note that the total heat transfer from the fin is proportional to its surface area. If the surface area at the fin tip is a negligible fraction of the total surface area, then it is reasonable to assume that $\theta'(L) = 0$.

  In this case, the temperature excess is $\theta(x) = \theta_0 \frac{\cosh(m(L-x))}{\cosh(mL)}$.

  **Check!** Instead of computing this from scratch (do that as well, later!), check that this indeed solves the DE as well as the boundary conditions $\theta(0) = \theta_0$ and $\theta'(L) = 0$. This should be a rather quick check!

- $\theta(L) = \theta_L$ (specified temperature at fin tip)

  In this case, the temperature excess is $\theta(x) = \frac{\theta_L \sinh(mx) + \theta_0 \sinh(m(L-x))}{\sinh(mL)}$.

  **Check!** Again, check that this indeed solves the DE as well as the boundary conditions $\theta(0) = \theta_0$ and $\theta(L) = \theta_L$. Once more, this should be a quick and pleasant check.

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