Example 170. Consider the polygonal region with vertices $(0,0)$, $(4,0)$, $(4,2)$, $(2,2)$, $(2,3)$, $(0,3)$. We wish to find the steady-state temperature distribution $u(x,y)$ within this region if the temperature is A between $(0,0)$, $(4,0)$, and B elsewhere on the boundary.

Spell out the resulting equations when we discretize this problem using a step size of $h=1$.

Solution. As before, we write $u_{m,n} = u(mh, nh)$. Make a sketch!

$$\begin{array}{ccccc} & B & & & \\ B & u_{1,2} & B & B & \\ B & u_{1,1} & u_{2,1} & u_{3,1} & B \\ & A & A & A & \end{array}$$

If we approximate $u_{xx} + u_{yy}$ by $\frac{1}{h^2}[u(x+h,y) + u(x-h,y) + u(x,y+h) + u(x,y-h) - 4u(x,y)]$ then, in terms of our lattice points, the equation $u_{xx} + u_{yy} = 0$ translates into

$$u_{m+1,n} + u_{m-1,n} + u_{m,n+1} + u_{m,n-1} - 4u_{m,n} = 0.$$

Spelling out these equation in matrix-vector form, we obtain:

$$\begin{bmatrix} -4 & 1 & 0 & 1 \\ 1 & -4 & 1 & 0 \\ 0 & 1 & -4 & 0 \\ 1 & 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \\ u_{1,2} \end{bmatrix} = \begin{bmatrix} -A-B \\ -A-B \\ -A-2B \\ -3B \end{bmatrix}$$

Comment. Note that, because of the way we discretize, it matters that there is a well-defined temperature at the boundary vertex $(2,2)$. For the other vertices, we don't need a well-defined temperature (and so it is not a problem that it is unclear what the temperature should be at $(0,0)$ or $(4,0)$ where it jumps from A to B).

Solving the Laplace equation inside a rectangle

One complication in solving the 2D Laplace equation inside a region R with Dirichlet boundary conditions is that the boundary of R is some curve (opposed to just two points in the 1D case).

A strategy to deal with complicated regions is to break them into simpler regions, such as rectangles or triangles.

Next, we demonstrate that we can fully solve the 2D Laplace equation in the case when R is a rectangle. In that case, the Dirichlet problem takes the form:

$$\begin{array}{rcll} u_{xx} + u_{yy} & = & 0 & \text{(PDE)} \\ u(x,0) & = & f_1(x) & \\ u(x,b) & = & f_2(x) & \text{(BC)} \\ u(0,y) & = & f_3(y) & \\ u(a,y) & = & f_4(y) & \end{array}$$

The first crucial observation is that we can break this problem into four parts:

$$\begin{array}{llll} u_{xx}+u_{yy}=0 & u_{xx}+u_{yy}=0 & u_{xx}+u_{yy}=0 & u_{xx}+u_{yy}=0 \\ u(x,0) = f_1(x) & u(x,0) = 0 & u(x,0) = 0 & u(x,0) = 0 \\ u(x,b) = 0 & u(x,b) = f_2(x) & u(x,b) = 0 & u(x,b) = 0 \\ u(0,y) = 0 & u(0,y) = 0 & u(0,y) = f_3(y) & u(0,y) = 0 \\ u(a,y) = 0 & u(a,y) = 0 & u(a,y) = 0 & u(a,y) = f_4(y) \end{array}$$

If we solve these four simpler Dirichlet problems, then the sum of the four solutions will solve the original Dirichlet problem.

As a consequence, it is no loss of generality to use homogeneous boundary conditions for three of the four sides. We illustrate how to solve this case in the next example. Our main tool is separation of variables, just as for the heat equation $u_t = k u_{xx}$.

Example 171. Find the unique solution $u(x,y)$ to:

$$\begin{aligned} &u_{xx} + u_{yy} = 0 && \text{(PDE)} \\ &u(x,0) = f(x) \\ &u(x,b) = 0 && \text{(BC)} \\ &u(0,y) = 0 \\ &u(a,y) = 0 \end{aligned}$$

Solution.

- We proceed as before and look for solutions $u(x,y) = X(x)Y(y)$ (**separation of variables**).
  Plugging into (PDE), we get $X''(x)Y(y) + X(x)Y''(y)$, and so $\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \text{const} =: -\lambda$.
  We thus have $X'' + \lambda X = 0$ and $Y'' - \lambda Y = 0$.
- From the last three (BC), we get $X(0) = 0$, $X(a) = 0$, $Y(b) = 0$.
  We ignore the first (inhomogeneous) condition for now.
- So X solves $X'' + \lambda X = 0$, $X(0) = 0$, $X(a) = 0$.
  From earlier, we know that, up to multiples, the only nonzero solutions of this eigenvalue problem are $X(x) = \sin(\frac{\pi n}{a} x)$ corresponding to $\lambda = (\frac{\pi n}{a})^2$, $n = 1, 2, 3\ldots$.
- On the other hand, Y solves $Y'' - \lambda Y = 0$, and hence $Y(y) = A e^{\sqrt{\lambda} y} + B e^{-\sqrt{\lambda} y}$.
  The condition $Y(b) = 0$ implies that $A e^{\sqrt{\lambda} b} + B e^{-\sqrt{\lambda} b} = 0$ so that $B = -A e^{2\sqrt{\lambda} b}$.
  Hence, $Y(y) = A\big(e^{\sqrt{\lambda} y} - e^{\sqrt{\lambda}(2b - y)}\big)$.
- Taken together, we have the solutions $u_n(x,y) = \sin(\frac{\pi n}{a} x)\Big(e^{\frac{\pi n}{a} y} - e^{\frac{\pi n}{a}(2b - y)}\Big)$ solving (PDE)+(BC), with the exception of $u(x,0) = f(x)$.
- We wish to combine these in such a way that $u(x,0) = f(x)$ holds as well.
  At $y = 0$, $u_n(x,0) = \sin(\frac{\pi n}{a} x)(1 - e^{2\pi n b/a})$. All of these are $2a$-periodic.
  Hence, we extend $f(x)$, which is only given on $(0,a)$, to an odd $2a$-periodic function (its Fourier sine series!). By making it odd, its Fourier series will only involve sine terms: $f(x) = \sum_{n=1}^{\infty} b_n \sin(\frac{\pi n}{a} x)$.
  Note that

  $$b_n = \frac{1}{a} \int_{-a}^{a} f(x) \sin\Big(\frac{n\pi x}{a}\Big) \mathrm{d}x = \frac{2}{a} \int_0^a f(x) \sin\Big(\frac{n\pi x}{a}\Big) \mathrm{d}x,$$

  where the first integral makes reference to the extension of $f(x)$ while the second integral only uses $f(x)$ on its original interval of definition.

Consequently, (PDE)+(BC) is solved by

$$u(x,y) = \sum_{n=1}^{\infty} \frac{b_n}{1 - e^{2\pi n b/a}} u_n(x,y) = \sum_{n=1}^{\infty} \frac{b_n}{1 - e^{2\pi n b/a}} \sin\Big(\frac{\pi n}{a} x\Big)\Big(e^{\frac{\pi n}{a} y} - e^{\frac{\pi n}{a}(2b - y)}\Big),$$

where

$$b_n = \frac{2}{a} \int_0^a f(x) \sin\Big(\frac{n\pi x}{a}\Big) \mathrm{d}x.$$

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Example 172. Find the unique solution $u(x,y)$ to:
$$\begin{aligned} u_{xx}+u_{yy} &= 0 \quad \text{(PDE)} \\ u(x,0) &= 1 \\ u(x,2) &= 0 \qquad \text{(BC)} \\ u(0,y) &= 0 \\ u(1,y) &= 0 \end{aligned}$$

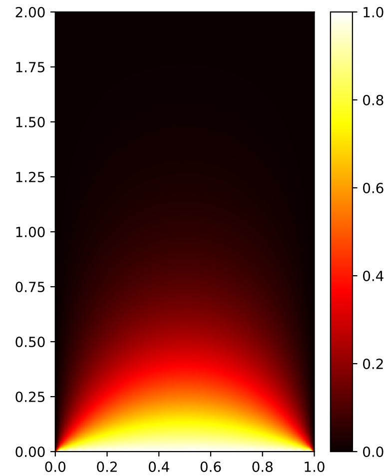
Solution. This is the special case of the previous example with $a=1$, $b=2$ and $f(x)=1$ for $x\in(0,1)$.

From Example 137, we know that $f(x)$ has the Fourier sine series

$$f(x)=\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty}\frac{4}{\pi n}\sin(n\pi x), \quad x\in(0,1).$$

Hence,

$$u(x,y)=\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty}\frac{4}{\pi n}\frac{1}{1-e^{4\pi n}}\sin(\pi n x)(e^{\pi n y}-e^{\pi n(4-y)}).$$



Comment. The temperature at the center is $u(\frac{1}{2},1)\approx 0.0549$ (only the first term of the infinite sum suffices for this estimate; the first three terms suffice for 9 digits of accuracy).

Example 173. Find the unique solution $u(x,y)$ to:
$$\begin{aligned} u_{xx}+u_{yy} &= 0 \quad \text{(PDE)} \\ u(x,0) &= 0 \\ u(x,2) &= 3 \qquad \text{(BC)} \\ u(0,y) &= 0 \\ u(1,y) &= 0 \end{aligned}$$

Solution. Instead of starting from scratch (homework exercise!), let us reuse our computations:
Let $v(x,y)=u(x,2-y)$. Then $v_{xx}+v_{yy}=0$, $v(x,0)=3$, $v(x,2)=0$, $v(0,y)=0$, $v(1,y)=0$.
Hence, it follows from the previous example that

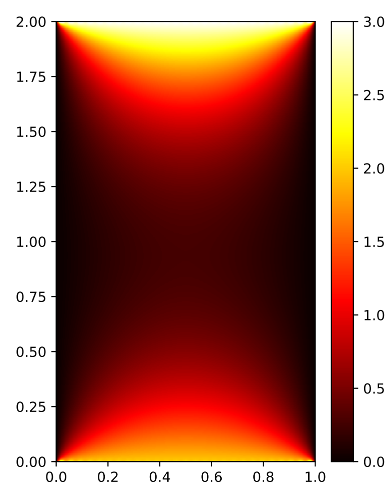
$$v(x,y)=3\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty}\frac{4}{\pi n}\frac{1}{1-e^{4\pi n}}\sin(\pi n x)(e^{\pi n y}-e^{\pi n(4-y)}).$$

Consequently,

$$u(x,y)=v(x,2-y)=3\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty}\frac{4}{\pi n}\frac{1}{1-e^{4\pi n}}\sin(\pi n x)(e^{\pi n(2-y)}-e^{\pi n(2+y)}).$$

Example 174. Find the unique solution $u(x,y)$ to:

$$\begin{aligned} u_{xx}+u_{yy} &= 0 \\ u(x,0)=2, \quad u(x,2) &= 3 \\ u(0,y)=0, \quad u(1,y) &= 0 \end{aligned}$$



Solution. Note that $u(x,\ y)$ is a combination of the solutions to the previous two examples!

$$u(x,y)=\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty}\frac{4}{\pi n}\frac{\sin(\pi n x)}{1-e^{4\pi n}}[2(e^{\pi n y}-e^{-\pi n(y-4)})+3(e^{\pi n(2-y)}-e^{\pi n(2+y)})].$$

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Example 175. Find the unique solution $u(x, y)$ to:

$$\begin{aligned} u_{xx} + u_{yy} &= 0 && \text{(PDE)} \\ u(x,0) &= 4\sin(\pi x) - 5\sin(3\pi x) \\ u(x,2) &= 0 && \text{(BC)} \\ u(0,y) &= 0 \\ u(3,y) &= 0 \end{aligned}$$

Solution.

- We look for solutions $u(x, y) = X(x)Y(y)$ (**separation of variables**).
  Plugging into (PDE), we get $X''(x)Y(y) + X(x)Y''(y)$, and so $\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \text{const}$.
  We thus have $X'' - \text{const}\, X = 0$ and $Y'' + \text{const}\, Y = 0$.

- From the last three (BC), we get $X(0) = 0$, $X(3) = 0$, $Y(2) = 0$.

- So X solves $X'' + \lambda X = 0$ (we choose $\lambda = -\text{const}$), $X(0) = 0$, $X(3) = 0$.
  From earlier (or do it!), we know that, up to multiples, the only nonzero solutions of this eigenvalue problem are $X(x) = \sin\left(\frac{1}{3}\pi n x\right)$ corresponding to $\lambda = \left(\frac{1}{3}\pi n\right)^2$, $n = 1, 2, 3 \ldots$.

- On the other hand, Y solves $Y'' - \lambda Y = 0$, and hence $Y(y) = A e^{\sqrt{\lambda} y} + B e^{-\sqrt{\lambda} y}$.
  The condition $Y(2) = 0$ implies that $A e^{2\sqrt{\lambda}} + B e^{-2\sqrt{\lambda}} = 0$ so that $B = -A e^{4\sqrt{\lambda}}$.
  Hence, $Y(y) = A(e^{\sqrt{\lambda} y} - e^{\sqrt{\lambda}(4-y)}) = A\left(e^{\frac{1}{3}\pi n y} - e^{\frac{1}{3}\pi n(4-y)}\right)$.

- Taken together, we have the solutions $u_n(x, y) = \sin\left(\frac{1}{3}\pi n x\right)\left(e^{\frac{1}{3}\pi n y} - e^{\frac{1}{3}\pi n(4-y)}\right)$ solving (PDE)+(BC), with the exception of $u(x, 0) = 4\sin(\pi x) - 5\sin(3\pi x)$.

- At $y = 0$, $u_n(x, 0) = \sin\left(\frac{1}{3}\pi n x\right)\left(1 - e^{\frac{4}{3}\pi n}\right)$.
  In particular, $u_3(x, 0) = \sin(\pi x)(1 - e^{4\pi})$ and $u_9(x, 0) = \sin(3\pi x)(1 - e^{12\pi})$.
  Hence, $4\sin(\pi x) - 5\sin(3\pi x) = \frac{4}{1 - e^{4\pi}} u_3(x, 0) - \frac{5}{1 - e^{12\pi}} u_9(x, 0)$. Therefore our overall solution is

$$\begin{aligned} u(x, y) &= \frac{4}{1 - e^{4\pi}} u_3(x, y) - \frac{5}{1 - e^{12\pi}} u_9(x, y) \\ &= \frac{4}{1 - e^{4\pi}} \sin(\pi x)(e^{\pi y} - e^{\pi(4-y)}) - \frac{5}{1 - e^{12\pi}} \sin(3\pi x)(e^{3\pi y} - e^{3\pi(4-y)}). \end{aligned}$$

Comment. Of course, in general, our inhomogeneous (BC) will be a function $f(x)$ that is not such an obvious combination of our special solutions $u_n(x, 0)$. In that case, we need to compute an appropriate Fourier expansion of $f(x)$ first (here, the Fourier sine series of $f(x)$).

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