Boundary value problems

Example 149. The **IVP** (initial value problem) $y'' + 4y = 0$, $y(0) = 0$, $y'(0) = 0$ has the unique solution $y(x) = 0$.

Initial value problems are often used when the problem depends on time. Then, $y(0)$ and $y'(0)$ describe the initial configuration at $t = 0$.

For problems which instead depend on spatial variables, such as position, it may be natural to specify values at positions on the boundary (for instance, if $y(x)$ describes the steady-state temperature of a rod at position x, we might know the temperature at the two end points).

The next example illustrates that such a boundary value problem (BVP) may or may not have a unique solution.

Example 150. Verify the following claims.

(a) The BVP $y'' + 4y = 0$, $y(0) = 0$, $y(1) = 0$ has the unique solution $y(x) = 0$.

(b) The BVP $y'' + \pi^2 y = 0$, $y(0) = 0$, $y(1) = 0$ is solved by $y(x) = B\sin(\pi x)$ for any value B.

Solution.

(a) We know that the general solution to the DE is $y(x) = A\cos(2x) + B\sin(2x)$. The boundary conditions imply $y(0) = A \overset{!}{=} 0$ and, using that $A = 0$, $y(1) = B\sin(2) \overset{!}{=} 0$ shows that $B = 0$ as well.

(b) This time, the general solution to the DE is $y(x) = A\cos(\pi x) + B\sin(\pi x)$. The boundary conditions imply $y(0) = A \overset{!}{=} 0$ and, using that $A = 0$, $y(1) = B\sin(\pi) \overset{!}{=} 0$. This second condition is true for every B.

It is therefore natural to ask: for which λ does the BVP $y'' + \lambda y = 0$, $y(0) = 0$, $y(L) = 0$ have nonzero solutions? (We assume that $L > 0$.)

Such solutions are called **eigenfunctions** and λ is the corresponding **eigenvalue**.

Remark. Compare that to our previous use of the term eigenvalue: given a matrix A, we asked: for which λ does $A\boldsymbol{v} - \lambda\boldsymbol{v} = 0$ have nonzero solutions $\boldsymbol{v}$? Such solutions were called eigenvectors and λ was the corresponding eigenvalue.

Example 151. Find all eigenfunctions and eigenvalues of $y'' + \lambda y = 0$, $y(0) = 0$, $y(L) = 0$.

Such a problem is called an **eigenvalue problem**.

Solution. The solutions of the DE look different in the cases $\lambda < 0$, $\lambda = 0$, $\lambda > 0$, so we consider them individually.

$\boldsymbol{\lambda = 0}$. Then $y(x) = Ax + B$ and $y(0) = y(L) = 0$ implies that $y(x) = 0$. No eigenfunction here.

$\boldsymbol{\lambda < 0}$. The roots of the characteristic polynomial are $\pm\sqrt{-\lambda}$. Writing $\rho = \sqrt{-\lambda}$, the general solution therefore is $y(x) = Ae^{\rho x} + Be^{-\rho x}$. $y(0) = A + B \overset{!}{=} 0$ implies $B = -A$. Using that, we get $y(L) = A(e^{\rho L} - e^{-\rho L}) \overset{!}{=} 0$. For eigenfunctions we need $A \neq 0$, so $e^{\rho L} = e^{-\rho L}$ which implies $\rho L = -\rho L$. This cannot happen since $\rho \neq 0$ and $L \neq 0$. Again, no eigenfunctions in this case.

$\boldsymbol{\lambda > 0}$. The roots of the characteristic polynomial are $\pm i\sqrt{\lambda}$. Writing $\rho = \sqrt{\lambda}$, the general solution thus is $y(x) = A\cos(\rho x) + B\sin(\rho x)$. $y(0) = A \overset{!}{=} 0$. Using that, $y(L) = B\sin(\rho L) \overset{!}{=} 0$. Since $B \neq 0$ for eigenfunctions, we need $\sin(\rho L) = 0$. This happens if $\rho L = n\pi$ for $n = 1, 2, 3, \ldots$ (since ρ and L are both positive, n must be positive as well). Equivalently, $\rho = \frac{n\pi}{L}$. Consequently, we find the eigenfunctions $y_n(x) = \sin\frac{n\pi x}{L}$, $n = 1, 2, 3, \ldots$, with eigenvalue $\lambda = \left(\frac{n\pi}{L}\right)^2$.

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Example 152. Suppose that a rod of length L is compressed by a force P (with ends being pinned [not clamped] down). We model the shape of the rod by a function $y(x)$ on some interval $[0, L]$. The theory of elasticity predicts that, under certain simplifying assumptions, y should satisfy $EIy'' + Py = 0,\ y(0) = 0,\ y(L) = 0$.

Here, EI is a constant modeling the inflexibility of the rod (E, known as Young's modulus, depends on the material, and I depends on the shape of cross-sections (it is the area moment of inertia)).

In other words, $y'' + \lambda y = 0,\ y(0) = 0,\ y(L) = 0$, with $\lambda = \frac{P}{EI}$.

The fact that there is no nonzero solution unless $\lambda = \left(\frac{\pi n}{L}\right)^2$ for some $n = 1, 2, 3, \ldots$, means that buckling can only occur if $P = \left(\frac{\pi n}{L}\right)^2 EI$. In particular, no buckling occurs for forces less than $\frac{\pi^2 EI}{L^2}$. This is known as the critical load (or Euler load) of the rod.

Comment. This is a very simplified model. In particular, it assumes that the deflections are small. (Technically, the buckled rod in our model is longer than L; of course, that's not the case in practice.)

https://en.wikipedia.org/wiki/Euler%27s_critical_load

Example 153. Find all eigenfunctions and eigenvalues of

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(3) = 0.$$

Solution. We distinguish three cases:

- $\boldsymbol{\lambda < 0}$. The characteristic roots are $\pm r = \pm\sqrt{-\lambda}$ and the general solution to the DE is $y(x) = Ae^{rx} + Be^{-rx}$. Then $y'(0) = Ar - Br = 0$ implies $B = A$, so that $y(3) = A(e^{3r} + e^{-3r})$. Since $e^{3r} + e^{-3r} > 0$, we see that $y(3) = 0$ only if $A = 0$. So there is no solution for $\lambda < 0$.
- $\boldsymbol{\lambda = 0}$. The general solution to the DE is $y(x) = A + Bx$. Then $y'(0) = 0$ implies $B = 0$, and it follows from $y(3) = A = 0$ that $\lambda = 0$ is not an eigenvalue.
- $\boldsymbol{\lambda > 0}$. The characteristic roots are $\pm i\sqrt{\lambda}$. So, with $r = \sqrt{\lambda}$, the general solution is $y(x) = A\cos(rx) + B\sin(rx)$. $y'(0) = Br = 0$ implies $B = 0$. Then $y(3) = A\cos(3r) = 0$. Note that $\cos(3r) = 0$ is true if and only if $3r = \frac{\pi}{2} + n\pi = \frac{(2n+1)\pi}{2}$ for some integer n. Since $r > 0$, we have $n \geqslant 0$. Correspondingly, $\lambda = r^2 = \left(\frac{(2n+1)\pi}{6}\right)^2$ and $y(x) = \cos\left(\frac{(2n+1)\pi}{6}x\right)$.

In summary, we have that the eigenvalues are $\lambda = \left(\frac{(2n+1)\pi}{6}\right)^2$, with $n = 0, 1, 2, \ldots$ with corresponding eigenfunctions $y(x) = \cos\left(\frac{(2n+1)\pi}{6}x\right)$.

Partial differential equations

The heat equation

We wish to describe one-dimensional heat flow.

Comment. If this sounds very specialized, it might help to know that the heat equation is also used, for instance, in probability (Brownian motion), financial math (Black-Scholes), or chemical processes (diffusion equation).

Let $u(x, t)$ describe the temperature at time t at position x.

If we model a heated rod of length L, then $x \in [0, L]$.

Notation. $u(x, t)$ depends on two variables. When taking derivatives, we will use the notations $u_t = \frac{\partial}{\partial t}u$ and $u_{xx} = \frac{\partial^2}{\partial x^2}u$ for first and higher derivatives.

Experience tells us that heat flows from warmer to cooler areas and has an averaging effect.

Make a sketch of some temperature profile $u(x,t)$ for fixed t.

As t increases, we expect maxima (where $u_{xx} < 0$) of that profile to flatten out (which means that $u_t < 0$); similarly, minima (where $u_{xx} > 0$) should go up (meaning that $u_t > 0$). The simplest relationship between u_t and u_{xx} which conforms with our expectation is $u_t = k u_{xx}$, with $k > 0$.

(heat equation)

$$u_t = k u_{xx}$$

Note that the heat equation is a linear and homogeneous **partial differential equation**.

In particular, the principle of superposition holds: if u_1 and u_2 solve the heat equation, then so does $c_1 u_1 + c_2 u_2$.

Higher dimensions. In higher dimensions, the heat equation takes the form $u_t = k(u_{xx} + u_{yy})$ or $u_t = k(u_{xx} + u_{yy} + u_{zz})$. The heat equation is often written as $u_t = k\Delta u$, where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplace operator you may know from Calculus III.

The Laplacian Δu is also often written as $\Delta u = \nabla^2 u$. The operator $\nabla = (\partial/\partial x, \partial/\partial y)$ is pronounced "nabla" (Greek for a certain harp) or "del" (Persian for heart), and ∇^2 is short for the inner product $\nabla \cdot \nabla$.

Example 154. Note that $u(x,t) = ax + b$ solves the heat equation.

Example 155. To get a feeling, let us find some other solutions to $u_t = u_{xx}$ (for starters, $k = 1$).

- For instance, $u(x,t) = e^t e^x$ is a solution.
  [Not a very interesting one for modeling heat flow because it increases exponentially in time.]
- ... to be continued ...
  Can you find further solutions?

Let us think about what is needed to describe a unique solution of the heat equation.

- **Initial condition** at $t = 0$: $\quad u(x,0) = f(x) \qquad \text{(IC)}$

  This specifies an initial temperature distribution at time $t = 0$.

- **Boundary condition** at $x = 0$ and $x = L$: $\qquad \text{(BC)}$

  Assuming that heat only enters/exits at the boundary (think of our rod as being insulated, except possibly at the two ends), we need some condition on the temperature at the ends. For instance:

  - $u(0,t) = A$, $u(L,t) = B$

    This models a rod where one end is kept at temperature A and the other end at temperature B.

  - $u_x(0,t) = u_x(L,t) = 0$

    This models a rod whose ends are insulated as well.

Under such assumptions, our physical intuition suggests that there should be a unique solution.

Important comment. We can always transform the case $u(0,t) = A$, $u(L,t) = B$ into $u(0,t) = u(L,t) = 0$ by using the fact that $u(t,x) = ax + b$ solves $u_t = k u_{xx}$. Can you spell this out?