Excursion: The Riemann hypothesis—A Millennium Prize Problem

The **Riemann zeta function** is defined by $\zeta(s) = \sum_{n \geqslant 1} \frac{1}{n^s}$.

Note that this series converges (for real s) if and only if $s > 1$.
The divergent series $\zeta(1)$ is the harmonic series, and $\zeta(p)$ is often called a p-series in Calculus II.

Example 148. Recall from Example 140 that using Fourier series, we were able to find that $\frac{\pi^2}{8} = \frac{1}{1} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$ Use this to derive $\zeta(2) = \sum_{n \geqslant 1} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Solution. If we split the sum into those terms where n is even and those where n is odd, then we get

$$\sum_{n \geqslant 1} \frac{1}{n^2} = \sum_{n \geqslant 1} \frac{1}{(2n)^2} + \sum_{n \geqslant 1} \frac{1}{(2n-1)^2} = \frac{1}{4} \sum_{n \geqslant 1} \frac{1}{n^2} + \frac{\pi^2}{8}.$$

If we write $x = \sum_{n \geqslant 1} \frac{1}{n^2}$, then this means that $x = \frac{1}{4}x + \frac{\pi^2}{8}$. We can then solve this equation to find $x = \frac{\pi^2}{6}$, which is what we wanted to derive.

Comment. Euler achieved worldwide fame in 1734 by discovering and proving that $\zeta(2) = \frac{\pi^2}{6}$ (as well as $\sum_{n \geqslant 1} \frac{1}{n^4} = \frac{\pi^4}{90}$ and similar formulas for $\zeta(6), \zeta(8), \ldots$).

Comment. On the other hand, no such evaluations are known for $\zeta(3), \zeta(5), \ldots$ and we don't even know (for sure) whether these are rational numbers. Nobody believes these to be rational numbers, but it was only in 1978 that Apéry proved that $\zeta(3)$ is not a rational number.

The Clay Mathematics Institute has offered 10^6 dollars each for the first correct solution to seven **Millennium Prize Problems**. Six of the seven problems remain open.

https://en.wikipedia.org/wiki/Millennium_Prize_Problems

Comment. Grigori Perelman solved the Poincaré conjecture in 2003 (but refused the prize money in 2010).
https://en.wikipedia.org/wiki/Poincaré_conjecture

The Riemann hypothesis is one of the seven Millennium Prize Problems and is concerned with the zeros of the Riemann zeta function $\zeta(s)$.

Recall that $\zeta(1)$ is the harmonic series, which diverges. For complex values of $s \neq 1$, there is a unique way to "analytically continue" the function $\zeta(s)$ from the definition $\zeta(s) = \sum_{n \geqslant 1} \frac{1}{n^s}$ which only works for $\operatorname{Re} s > 1$.

It is then "easy" to see that $\zeta(-2) = 0$, $\zeta(-4) = 0$, ... These are called the trivial zeros of $\zeta(s)$.

The **Riemann hypothesis** claims that all other zeros of $\zeta(s)$ lie on the (vertical) line $s = \frac{1}{2} + ai$ (where a is real).

A proof of this conjecture (checked for the first 10,000,000,000 zeroes) is worth $1,000,000.

http://www.claymath.org/millennium-problems/riemann-hypothesis

The reason for caring about the zeros is that they are intimately tied to the distribution of primes. The prime number theorem states that, up to x, there are about $x / \ln(x)$ many primes. The Riemann hypothesis gives very precise error estimates for an improved prime number theorem (using a function more complicated than the logarithm).

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The connection to primes. Here's a vague indication that $\zeta(s)$ is intimately connected to prime numbers:

$$\begin{aligned}\zeta(s)=\sum_{n\geqslant 1}\frac{1}{n^s} &= \left(1+\frac{1}{2^s}+\frac{1}{2^{2s}}+\ldots\right)\left(1+\frac{1}{3^s}+\frac{1}{3^{2s}}+\ldots\right)\left(1+\frac{1}{5^s}+\frac{1}{5^{2s}}+\ldots\right)\cdots \\ &= \frac{1}{1-2^{-s}}\frac{1}{1-3^{-s}}\frac{1}{1-5^{-s}}\cdots \\ &= \prod_{p\text{ prime}}\frac{1}{1-p^{-s}}\end{aligned}$$

To see that the second equality holds, imagine multiplying out the right-hand side. For instance, we get the term $\frac{1}{2^s}\cdot\frac{1}{3^{4s}}\cdot\frac{1}{7^s}=\frac{1}{(2\cdot 3^4\cdot 7)^s}$. This matches the term $\frac{1}{n^s}$ on the left-hand side for $n=2\cdot 3^4\cdot 7$. Since every positive integer n has a unique factorization into prime factors, this matching is one-to-one.

The final infinite product is called the Euler product for the zeta function.

If the Riemann hypothesis was true, then we would be better able to estimate the number $\pi(x)$ of primes $p\leqslant x$. More generally, certain statements about the zeta function can be translated to statements about primes. For instance, the (non-obvious!) fact that $\zeta(s)$ has no zeros for $\operatorname{Re} s=1$ implies the prime number theorem.

http://www-users.math.umn.edu/~garrett/m/v/pnt.pdf