Fourier cosine series and Fourier sine series

Suppose we have a function $f(t)$ which is defined on a finite interval $[0, L]$. Depending on the kind of application, we can extend $f(t)$ to a periodic function in three natural ways; in each case, we can then compute a Fourier series for $f(t)$ (which will agree with $f(t)$ on $[0, L]$).

Comment. Here, we do not worry about the definition of $f(t)$ at specific individual points like $t=0$ and $t=L$, or at jump discontinuities. Recall that, at a discontinuity, a Fourier series takes the average value.

(a) We can extend $f(t)$ to an L-periodic function.

In that case, we obtain the Fourier series $f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{2\pi nt}{L}\right)+b_n\sin\left(\frac{2\pi nt}{L}\right)\right)$.

(b) We can extend $f(t)$ to an even $2L$-periodic function.

In that case, we obtain the **Fourier cosine series** $f(t)=\frac{\tilde{a}_0}{2}+\sum_{n=1}^{\infty}\tilde{a}_n\cos\left(\frac{\pi nt}{L}\right)$.

(c) We can extend $f(t)$ to an odd $2L$-periodic function.

In that case, we obtain the **Fourier sine series** $f(t)=\sum_{n=1}^{\infty}\tilde{b}_n\sin\left(\frac{\pi nt}{L}\right)$.

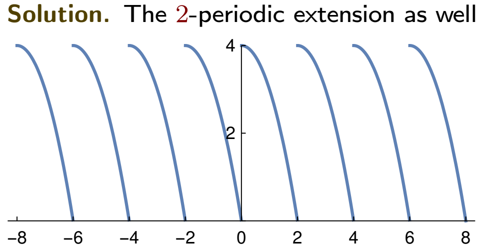
Example 141. Consider the function $f(t)=4-t^2$, defined for $t\in[0,2]$.

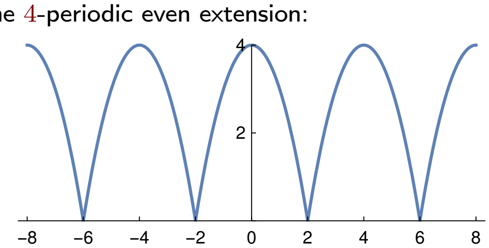
(a) Sketch the 2-periodic extension of $f(t)$.

(b) Sketch the 4-periodic even extension of $f(t)$.

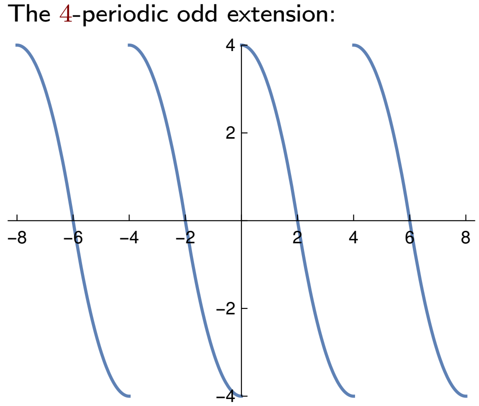
(c) Sketch the 4-periodic odd extension of $f(t)$.

Solution. The 2-periodic extension as well as the 4-periodic even extension:





The 4-periodic odd extension:



Example 142. As in the previous example, consider the function $f(t) = 4 - t^2$, defined for $t \in [0, 2]$.

(a) Let $F(t)$ be the Fourier series of $f(t)$ (meaning the 2-periodic extension of $f(t)$). Determine $F(2)$, $F\left(\frac{5}{2}\right)$ and $F\left(-\frac{1}{2}\right)$.

(b) Let $G(t)$ be the Fourier cosine series of $f(t)$. Determine $G(2)$, $G\left(\frac{5}{2}\right)$ and $G\left(-\frac{1}{2}\right)$.

(c) Let $H(t)$ be the Fourier sine series of $f(t)$. Determine $H(2)$, $H\left(\frac{5}{2}\right)$ and $H\left(-\frac{1}{2}\right)$.

Solution.

(a) Note that the extension of $f(t)$ has discontinuities at $\ldots, -2, 0, 2, 4, \ldots$ (see plot in previous example) and recall that the Fourier series takes average values at these discontinuities:

$F(2) = \frac{1}{2}(F(2^-) + F(2^+)) = \frac{1}{2}(0 + 4) = 2$

$F\left(\frac{5}{2}\right) = F\left(\frac{5}{2} - 2\right) = f\left(\frac{1}{2}\right) = \frac{15}{4}$

$F\left(-\frac{1}{2}\right) = F\left(-\frac{1}{2} + 2\right) = f\left(\frac{3}{2}\right) = \frac{7}{4}$

(b) $G(2) = f(2) = 0$ (see plot!)

[Note that $G(2^+) = G(2^+ - 4) = G(-2^+) = G(2^-)$ where we used that G is even in the last step; in fact, we can show like this that the Fourier cosine series of a continuous function is always continuous.]

$G\left(\frac{5}{2}\right) = G\left(\frac{5}{2} - 4\right) = G\left(-\frac{3}{2}\right) = f\left(\frac{3}{2}\right) = \frac{7}{4}$

$G\left(-\frac{1}{2}\right) = f\left(\frac{1}{2}\right) = \frac{15}{4}$

(c) $H(2) = \frac{1}{2}(f(2^-) - f(2^-)) = 0$ (see plot!)

[Note that $H(2^+) = H(2^+ - 4) = H(-2^+) = -H(2^-)$ where we used that H is odd in the last step; in fact, we can show like this that the Fourier sine series of a continuous function is always 0 at the jumps.]

$H\left(\frac{5}{2}\right) = H\left(\frac{5}{2} - 4\right) = H\left(-\frac{3}{2}\right) = -f\left(\frac{3}{2}\right) = -\frac{7}{4}$

$H\left(-\frac{1}{2}\right) = -f\left(\frac{1}{2}\right) = -\frac{15}{4}$

Fourier series and linear differential equations

In the following examples, we consider inhomogeneous linear DEs $p(D)y = F(t)$ where $F(t)$ is a periodic function that can be expressed as a Fourier series. We first review the notion of **resonance** (and how to predict it) and then solve such DEs.

Context. Recall that the inhomogeneous DE $my'' + ky = F(t)$ describes, for instance, the motion of a mass m on a spring with spring constant k under the influence of an external force $F(t)$. (Note how each term in the DE corresponds to a force: $my'' = ma$ from Newton's second law, ky from Hooke's law for springs, and $F(t)$ the external force.)

Example 143. Consider the linear DE $my'' + ky = \cos(\omega t)$. For which (external) **frequencies** $\omega > 0$ does **resonance** occur?

Solution. The characteristic roots (the roots of $p(D) = mD^2 + k$) are $\pm i\sqrt{k/m}$. Correspondingly, the solutions of the homogeneous equation $my'' + ky = 0$ are combinations of $\cos(\omega_0 t)$ and $\sin(\omega_0 t)$, where $\omega_0 = \sqrt{k/m}$ (ω_0 is called the **natural frequency** of the DE). Resonance occurs in the case $\omega = \omega_0$ when the external frequency matches the natural frequency.

Review. If $\omega \neq \omega_0$ (overlapping roots), then there is particular solution of the form $y_p(t) = A\cos(\omega t) + B\sin(\omega t)$ (for specific values of A and B). The general solution is $y(t) = A\cos(\omega t) + B\sin(\omega t) + C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)$, which is a bounded function of t. In contrast, if $\omega = \omega_0$, then the general solution is $y(t) = (C_1 + At)\cos(\omega_0 t) + (C_2 + Bt)\sin(\omega_0 t)$ and this function is unbounded.

Example 144. A mass-spring system is described by the DE $2y'' + 32y = \sum_{n=1}^{\infty} \frac{\cos(n\omega t)}{n^2 + 1}$.

For which ω does resonance occur?

Solution. The roots of $p(D) = 2D^2 + 32$ are $\pm 4i$, so that that the natural frequency is 4. Resonance therefore occurs if 4 equals $n\omega$ for some $n \in \{1, 2, 3, \ldots\}$. Equivalently, resonance occurs if $\omega = 4/n$ for some $n \in \{1, 2, 3, \ldots\}$.

Example 145. A mass-spring system is described by the DE $my'' + y = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left(\frac{nt}{3}\right)$.

For which m does resonance occur?

Solution. The roots of $p(D) = mD^2 + 1$ are $\pm i/\sqrt{m}$, so that the natural frequency is $1/\sqrt{m}$. Resonance therefore occurs if $1/\sqrt{m} = n/3$ for some $n \in \{1, 2, 3, \ldots\}$. Equivalently, resonance occurs if $m = 9/n^2$ for some $n \in \{1, 2, 3, \ldots\}$.

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