Fourier series with general period

There is nothing special about 2π-periodic functions considered before (except that $\cos(t)$ and $\sin(t)$ have fundamental period 2π). Note that $\cos(\pi t/L)$ and $\sin(\pi t/L)$ have period $2L$.

Theorem 136. Every* $2L$-periodic function f can be written as a **Fourier series**

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi t}{L}\right)+b_n\sin\left(\frac{n\pi t}{L}\right)\right).$$

Technical detail*: f needs to be, e.g., piecewise smooth.
Also, if t is a discontinuity, then the Fourier series converges to the average $\frac{f(t^-)+f(t^+)}{2}$.

The **Fourier coefficients** a_n, b_n are unique and can be computed as

$$a_n=\frac{1}{L}\int_{-L}^{L}f(t)\cos\left(\frac{n\pi t}{L}\right)\mathrm{d}t,\qquad b_n=\frac{1}{L}\int_{-L}^{L}f(t)\sin\left(\frac{n\pi t}{L}\right)\mathrm{d}t.$$

Comment. This follows from Theorem 131 because, if $f(t)$ has period $2L$, then $\tilde{f}(t):=f\left(\frac{L}{\pi}t\right)$ has period 2π.

Example 137. Find the Fourier series of the 2-periodic function $g(t)=\begin{cases}-1 & \text{for } t\in(-1,0)\\ +1 & \text{for } t\in(0,1)\\ 0 & \text{for } t=-1,0,1\end{cases}$.

Solution. Instead of computing from scratch, we can use the fact that $g(t)=f(\pi t)$, with f as in the previous example, to get $g(t)=f(\pi t)=\sum_{\substack{n=1\\ n\text{ odd}}}^{\infty}\frac{4}{\pi n}\sin(n\pi t)$.

Theorem 138. If $f(t)$ is **continuous** and $f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi t}{L}\right)+b_n\sin\left(\frac{n\pi t}{L}\right)\right)$, then* $f'(t)=\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}b_n\cos\left(\frac{n\pi t}{L}\right)-\frac{n\pi}{L}a_n\sin\left(\frac{n\pi t}{L}\right)\right)$ (i.e., we can differentiate termwise).

Technical detail*: f' needs to be, e.g., piecewise smooth (so that it has a Fourier series itself).

Caution! We cannot simply differentiate termwise if $f(t)$ is lacking continuity. See the next example.

Comment. On the other hand, we can integrate termwise (going from the Fourier series of $f'=g$ to the Fourier series of $f=\int g$ because the latter will be continuous). This is illustrated in the example after the next.

Example 139. (caution!) The function $g(t)=\sum_{n\text{ odd}}\frac{4}{\pi n}\sin(n\pi t)$ from Example 137 is not continuous. For all values, except the discontinuities, we have $g'(t)=0$. On the other hand, differentiating the Fourier series termwise, results in $4\sum_{n\text{ odd}}\cos(n\pi t)$, which diverges for most values of t (that's easy to check for $t=0$). This illustrates that we cannot apply Theorem 138 because $g(t)$ is lacking continuity.

[The issues we are facing here can be fixed by generalizing the notion of function to distributions. (Maybe you have heard of the Dirac delta "function".)]

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Example 140. Let $h(t)$ be the 2-periodic function with $h(t) = |t|$ for $t \in [-1, 1]$. Compute the Fourier series of $h(t)$.

Solution. We could just use the integral formulas to compute a_n and b_n. Since $h(t)$ is even (plot it!), we will find that $b_n = 0$. Computing a_n is left as an exercise.

Solution. Note that $h(t) = \begin{cases} -t & \text{for } t \in (-1, 0) \\ +t & \text{for } t \in (0, 1) \end{cases}$ is continuous and $h'(t) = g(t)$, with $g(t)$ as in Example 137. Hence, we can apply Theorem 138 to conclude

$$h'(t) = g(t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \sin(n\pi t) \quad \Longrightarrow \quad h(t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \left(-\frac{1}{\pi n}\right) \cos(n\pi t) + C,$$

where $C = \frac{a_0}{2} = \frac{1}{2} \int_{-1}^{1} h(t) \mathrm{d}t = \frac{1}{2}$ is the constant of integration. Thus, $h(t) = \frac{1}{2} - \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi^2 n^2} \cos(n\pi t)$.

Remark. Note that $t = 0$ in the last Fourier series, gives us $\frac{\pi^2}{8} = \frac{1}{1} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$. As an exercise, you can try to find from here the fact that $\sum_{n \geqslant 1} \frac{1}{n^2} = \frac{\pi^2}{6}$. Similarly, we can use Fourier series to find that $\sum_{n \geqslant 1} \frac{1}{n^4} = \frac{\pi^4}{90}$.
Just for fun. These are the values $\zeta(2)$ and $\zeta(4)$ of the Riemann zeta function $\zeta(s)$. No such evaluations are known for $\zeta(3), \zeta(5), \ldots$ and we don't even know (for sure) whether these are rational numbers. Nobody believes these to be rational numbers, but it was only in 1978 that Apéry proved that $\zeta(3)$ is not a rational number.

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