Power series of familiar functions

(Unless we specify otherwise, power series are meant to be about $x=0$.)

Example 126. The **hyperbolic cosine** $\cosh(x)$ is defined to be the even part of e^x. In other words, $\cosh(x)=\frac{1}{2}(e^x+e^{-x})$. Determine its power series.

Solution. It follows from $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$ that $\cosh(x)=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$.

Comment. Note that $\cosh(ix)=\cos(x)$ (because $\cos(x)=\frac{1}{2}(e^{ix}+e^{-ix})$).

Comment. The hyperbolic sine $\sinh(x)=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}$ is similarly defined to be the odd part of e^x.

Example 127. Determine a power series for $\frac{1}{1+x^2}$.

Solution. Replace x with $-x^2$ in $\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$ (geometric series!) to get $\frac{1}{1+x^2}=\sum_{n=0}^{\infty}(-1)^n x^{2n}$.

Example 128. Determine a power series for $\arctan(x)$.

Solution. Recall that $\arctan(x)=\int\frac{\mathrm{d}x}{1+x^2}+C$. Hence, we need to integrate $\frac{1}{1+x^2}=\sum_{n=0}^{\infty}(-1)^n x^{2n}$.

It follows that $\arctan(x)=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}+C$. Since $\arctan(0)=0$, we conclude that $C=0$.

Example 129. (error function) Determine a power series for $\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\mathrm{d}t$.

Solution. It follows from $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$ that $e^{-t^2}=\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{n!}$.

Integrating, we obtain $\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\mathrm{d}t=\frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{n!(2n+1)}$.

Example 130. Determine the first several terms (up to x^5) in the power series of $\tan(x)$.

Solution. Observe that $y(x)=\tan(x)$ is the unique solution to the IVP $y'=1+y^2$, $y(0)=0$.

We can therefore proceed to determine the first few power series coefficients as we did earlier.

That is, we plug $y=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\ldots$ into the DE. Note that $y(0)=0$ means $a_0=0$.

$y'=a_1+2a_2x+3a_3x^2+4a_4x^3+5a_5x^4+\ldots$

$1+y^2=1+(a_1x+a_2x^2+a_3x^3+\ldots)^2=1+a_1^2x^2+(2a_1a_2)x^3+(2a_1a_3+a_2^2)x^4+\ldots$

Comparing coefficients, we find: $a_1=1$, $2a_2=0$, $3a_3=a_1^2$, $4a_4=2a_1a_2$, $5a_5=2a_1a_3+a_2^2$.

Solving for $a_2,a_3,\ldots$, we conclude that $\tan(x)=x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}+\ldots$

Comment. The fact that $\tan(x)$ is an odd function translates into $a_n=0$ when n is even. If we had realized that at the beginning, our computation would have been simplified.

Advanced comment. The full power series is $\tan(x)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_{2n}}{(2n)!}x^{2n-1}$.

Here, the numbers B_{2n} are (rather mysterious) rational numbers known as **Bernoulli numbers**.

The radius of convergence is $\pi/2$. Note that this is not at all obvious from the DE $y'=1+y^2$. This illustrates the fact that nonlinear DEs are much more complicated than linear ones. (There is no analog of Theorem 114.)

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Fourier series

The following amazing fact is saying that any 2π-periodic function can be written as a sum of cosines and sines.

Advertisement. In Linear Algebra II, we will see the following natural way to look at Fourier series: the functions 1, $\cos(t)$, $\sin(t)$, $\cos(2t)$, $\sin(2t)$, ... are orthogonal to each other (for that to make sense, we need to think of functions as vectors and introduce a natural inner product). In fact, they form an orthogonal basis for the space of piecewise smooth functions. In that setting, the formulas for the coefficients a_n and b_n are nothing but the usual projection formulas for orthogonal projection onto a single vector.

Theorem 131. Every* 2π-periodic function f can be written as a **Fourier series**

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nt) + b_n \sin(nt)).$$

Technical detail*: f needs to be, e.g., piecewise smooth.
Also, if t is a discontinuity of f, then the Fourier series converges to the average $\frac{f(t^-) + f(t^+)}{2}$.

The **Fourier coefficients** a_n, b_n are unique and can be computed as

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\mathrm{d}t, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\mathrm{d}t.$$

Comment. Another common way to write Fourier series is $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$.
These two ways are equivalent; we can convert between them using Euler's identity $e^{int} = \cos(nt) + i\sin(nt)$.

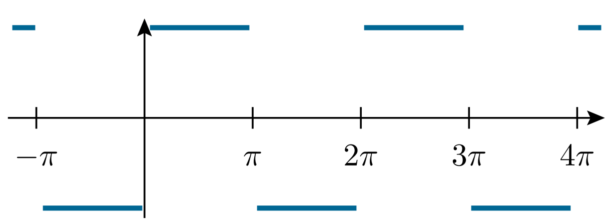
Definition 132. Let $L > 0$. $f(t)$ is ***L*-periodic** if $f(t+L) = f(t)$ for all t. The smallest such L is called the **(fundamental) period** of f.

Example 133. The fundamental period of $\cos(nt)$ is $2\pi/n$.

Example 134. The trigonometric functions $\cos(nt)$ and $\sin(nt)$ are 2π-periodic for every integer n. And so are their linear combinations. (Thus, 2π-periodic functions form a vector space!)

Example 135. Find the Fourier series of the 2π-periodic function $f(t)$ defined by

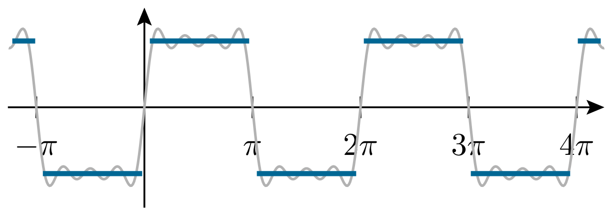
$$f(t) = \begin{cases} -1, & \text{for } t \in (-\pi, 0), \\ +1, & \text{for } t \in (0, \pi), \\ 0, & \text{for } t = -\pi, 0, \pi. \end{cases}$$



Solution. We compute $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\mathrm{d}t = 0$, as well as

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\mathrm{d}t = \frac{1}{\pi}\left[-\int_{-\pi}^{0}\cos(nt)\mathrm{d}t + \int_{0}^{\pi}\cos(nt)\mathrm{d}t\right] = 0 \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\mathrm{d}t = \frac{1}{\pi}\left[-\int_{-\pi}^{0}\sin(nt)\mathrm{d}t + \int_{0}^{\pi}\sin(nt)\mathrm{d}t\right] = \frac{2}{\pi n}[1-\cos(n\pi)] \\ &= \frac{2}{\pi n}[1-(-1)^n] = \begin{cases} \frac{4}{\pi n} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}. \end{aligned}$$

In conclusion, $f(t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n}\sin(nt) = \frac{4}{\pi}\left(\sin(t) + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t) + \ldots\right)$.



Observation. The coefficients a_n are zero for all n if and only if $f(t)$ is odd.

Comment. The value of $f(t)$ for $t = -\pi, 0, \pi$ is irrelevant to the computation of the Fourier series. They are chosen so that $f(t)$ is equal to the Fourier series for all t (recall that, at a jump discontinuity t, the Fourier series converges to the average $\frac{f(t^-)+f(t^+)}{2}$).

Comment. Plot the (sum of the) first few terms of the Fourier series. What do you observe? The "overshooting" is known as the **Gibbs phenomenon**: https://en.wikipedia.org/wiki/Gibbs_phenomenon

Comment. Set $t = \frac{\pi}{2}$ in the Fourier series we just computed, to get Leibniz' series $\pi = 4[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots]$. For such an alternating series, the error made by stopping at the term $1/n$ is on the order of $1/n$. To compute the 768 digits of π to get to the Feynman point ($3.14159265\ldots721134999999\ldots$), we would (roughly) need $1/n < 10^{-768}$, or $n > 10^{768}$. That's a lot of terms! (Roger Penrose, for instance, estimates that there are about 10^{80} atoms in the observable universe.)

Remark. Convergence of such series is not completely obvious! Recall, for instance, that the (odd part of) the harmonic series $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$ diverges. (On the other hand, do you remember the alternating sign test from Calculus II?)

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