Review. Theorem 114: If x_0 is an ordinary point of a linear IVP, then it is guaranteed to have a power series solution $y(x)=\sum_{n=0}^{\infty} a_n(x-x_0)^n$.

Moreover, its radius of convergence is at least the distance between x_0 and the closest singular point.

Example 118. Find a minimum value for the radius of convergence of a power series solution to $(x^2+4)y''-3xy'+\frac{1}{x+1}y=0$ at $x=2$.

Solution. The singular points are $x=\pm 2i,-1$. Hence, $x=2$ is an ordinary point of the DE and the distance to the nearest singular point is $|2-2i|=\sqrt{2^2+2^2}=\sqrt{8}$ (the distances are $|2-(-1)|=3$, $|2-(\pm 2i)|=\sqrt{8}$). By Theorem 114, the DE has power series solutions about $x=2$ with radius of convergence at least $\sqrt{8}$.

Example 119. (caution!) Theorem 114 only holds for linear DEs! For nonlinear DEs, it is very hard to predict whether there is a power series solution and what its radius of convergence is. Consider, for instance, the nonlinear DE $y'-y^2=0$.

Its coefficients have no singularities. A solution to this DE is $y(x)=\frac{1}{1-x}=\sum_{n=0}^{\infty} x^n$ (see Example 122), which clearly has a problem at $x=1$ (the radius of convergence is 1).

On the other hand. $y(x)$ also solves the linear DE $(1-x)y'-y=0$ (or, even simpler, the order 0 "differential" equation $(1-x)y=1$). Note how the DE has the singular point $x=1$. Theorem 114 then allows us to predict that $y(x)$ must have a power series with radius of convergence at least 1.

Example 120. (Bessel functions) Consider the DE $x^2y''+xy'+x^2y=0$. Derive a recursive description of a power series solutions $y(x)$ at $x=0$.

Caution! Note that $x=0$ is a singular point (the only) of the DE. Theorem 114 therefore does not guarantee a basis of power series solutions. [However, $x=0$ is what is called a **regular singular point**; for these, we are guaranteed one power series solution, as well as additional solutions expressed using logarithms and power series.]

Comment. We could divide the DE by x (but that wouldn't really change the computations below). The reason for not dividing that x is that this DE is the special case $\alpha=0$ of the **Bessel equation** $x^2y''+xy'+(x^2-\alpha^2)y=0$ (for which no such dividing is possible).

Solution. (plug in power series) Let us spell out power series for x^2y, xy', x^2y'' starting with $y(x)=\sum_{n=0}^{\infty} a_nx^n$:

$$x^2y(x)=\sum_{n=0}^{\infty} a_nx^{n+2}=\sum_{n=2}^{\infty} a_{n-2}x^n$$

$$xy'(x)=\sum_{n=1}^{\infty} na_nx^n \qquad \text{(because } y'(x)=\sum_{n=1}^{\infty} na_nx^{n-1}\text{)}$$

$$x^2y''(x)=\sum_{n=2}^{\infty} n(n-1)a_nx^n \qquad \text{(because } y''(x)=\sum_{n=2}^{\infty} n(n-1)a_nx^{n-2}\text{)}$$

Hence, the DE becomes $\sum_{n=2}^{\infty} n(n-1)a_nx^n+\sum_{n=1}^{\infty} na_nx^n+\sum_{n=2}^{\infty} a_{n-2}x^n=0$. We compare coefficients of x^n:

- $n=1$: $a_1=0$
- $n\geqslant 2$: $n(n-1)a_n+na_n+a_{n-2}=0$, which simplifies to $n^2a_n=-a_{n-2}$.
  It follows that $a_{2n}=\frac{(-1)^n}{4^n n!^2}a_0$ and $a_{2n+1}=0$.

Observation. The fact that we found $a_1=0$ reflects the fact that we cannot represent the general solution through power series alone.

Comment. If $a_0=1$, the function we found is a **Bessel function** and denoted as $J_0(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!^2}\left(\frac{x}{2}\right)^{2n}$.

The more general Bessel functions $J_\alpha(x)$ are solutions to the DE $x^2y''+xy'+(x^2-\alpha^2)y=0$.

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Example 121. (caution!) Consider the linear DE $x^2 y' = y - x$. Does it have a convergent power series solution at $x = 0$?

Important note. The DE $x^2 y' = y - x$ has the singular point $x = 0$. Hence, Theorem 114 does not apply.

Advanced. Moreover, in contrast to the previous example, $x = 0$ is not a **regular singular point**. Indeed, as we see below, there is no power series solution of the DE at all.

Solution. Let us look for a power series solution $y(x) = \sum_{n=0}^{\infty} a_n x^n$.

$$x^2 y'(x) = x^2 \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n+1} = \sum_{n=2}^{\infty} (n-1) a_{n-1} x^n$$

Hence, $x^2 y' = y - x$ becomes $\sum_{n=2}^{\infty} (n-1) a_{n-1} x^n = \sum_{n=0}^{\infty} a_n x^n - x$. We compare coefficients of x^n:

- $n = 0$: $a_0 = 0$.
- $n = 1$: $0 = a_1 - 1$, so that $a_1 = 1$.
- $n \geqslant 2$: $(n-1) a_{n-1} = a_n$, from which it follows that $a_n = (n-1) a_{n-1} = (n-1)(n-2) a_{n-2} = \cdots = (n-1)! a_1 = (n-1)!$.

Hence the DE has the "formal" power series solution $y(x) = \sum_{n=1}^{\infty} (n-1)! x^n$.

However, that series is divergent for all $x \neq 0$; that is, the radius of convergence is 0.

Inverses of power series

Example 122. (geometric series) $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

Why? If $y(x) = \sum_{n=0}^{\infty} x^n$, then $xy = y - 1$ (write down the power series for both sides!). Hence, $y = \frac{1}{1-x}$.

Alternatively, start with $y = \frac{1}{1-x}$ and note that y solves the order 0 "differential" (inhomogeneous) equation $(1-x) y = 1$. We can then determine a power series solution as we did in Example 110 to find $y = \sum_{n=0}^{\infty} x^n$.

Example 123. (extra) Determine a power series for $\ln(x)$ around $x = 1$.

Solution. This is equivalent to finding a power series for $\ln(x+1)$ around $x = 0$ (see the final step).

Observe that $\ln(x+1) = \int \frac{\mathrm{d}x}{1+x} + C$ and that $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$.

Integrating, $\ln(x+1) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C$. Since $\ln(1) = 0$, we conclude that $C = 0$.

Finally, $\ln(x+1) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$ is equivalent to $\ln(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} (x-1)^{n+1}$.

Comment. Choosing $x = 2$ in $\ln(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} (x-1)^{n+1}$ results in $\ln(2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$.

The latter is the alternating harmonic sum.

Can you see from the series for $\ln(x)$ why the harmonic sum $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ diverges?

Example 124. Derive a recursive description of the power series for $y(x) = \frac{1}{1 - x - x^2}$.

Solution. Note that $y(x)$ satisfies the "differential" equation $(1 - x - x^2)y = 1$ of order 0 (as such, we need 0 initial conditions). We can therefore determine a power series solution as we did in Example 110:

Write $y(x) = \sum_{n=0}^{\infty} a_n x^n$. Then

$$1 = (1 - x - x^2)\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+2}$$
$$= \sum_{n=0}^{\infty} a_n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n - \sum_{n=2}^{\infty} a_{n-2} x^n.$$

We compare coefficients of x^n:

- $n = 0$: $1 = a_0$.
- $n = 1$: $0 = a_1 - a_0$, so that $a_1 = a_0 = 1$.
- $n \geqslant 2$: $0 = a_n - a_{n-1} - a_{n-2}$ or, equivalently, $a_n = a_{n-1} + a_{n-2}$.

This is the recursive description of the Fibonacci numbers F_n! In particular $a_n = F_n$.

The first few terms. $\frac{1}{1 - x - x^2} = 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + 13x^6 + \ldots$

Comment. The function $y(x)$ is said to be a **generating function** for the Fibonacci numbers.

Challenge. Can you rederive Binet's formula from partial fractions and the geometric series?

Example 125. (HW) Derive a recursive description of the power series for $y(x) = \frac{1 + 7x}{1 - x - 2x^2}$.

Solution. Write $y(x) = \sum_{n=0}^{\infty} a_n x^n$. Then

$$1 + 7x = (1 - x - 2x^2)\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+1} - 2\sum_{n=0}^{\infty} a_n x^{n+2}$$
$$= \sum_{n=0}^{\infty} a_n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n - 2\sum_{n=2}^{\infty} a_{n-2} x^n.$$

We compare coefficients of x^n:

- $n = 0$: $1 = a_0$.
- $n = 1$: $7 = a_1 - a_0$, so that $a_1 = 7 + a_0 = 8$.
- $n \geqslant 2$: $0 = a_n - a_{n-1} - 2a_{n-2}$.

If we prefer, we can rewrite the final recurrence as $a_{n+2} - a_{n+1} - 2a_n = 0$ for $n \geqslant 0$. The initial conditions are $a_0 = 1$, $a_1 = 8$.

Comment. In terms of the recurrence operator N, the recurrence is $(N^2 - N - 2)a_n = 0$.

Comment. As in Example 55, we can solve this recurrence and obtain a Binet-like formula for a_n. In this particular case, we find $a_n = 3 \cdot 2^n - 2(-1)^n$.