Power series solutions to linear DEs

Note how in the last two examples the "plug in power series" approach was complicated by the fact that the DE was not linear (we had to expand y^2 as well as $\cos(x+y)$, respectively).

For linear DEs, this complication does not arise and we can readily determine the complete power series expansion of analytic solutions (with a recursive description of the coefficients).

Example 110. (Airy equation, part II) Let $y(x)$ be the unique solution to the IVP $y''=xy$, $y(0)=a$, $y'(0)=b$. Determine the power series of $y(x)$.

Solution. (plug in power series) Let us spell out the power series for y, y', y'' and xy:

$$y(x)=\sum_{n=0}^{\infty} a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty} n a_n x^{n-1}=\sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

$$xy(x)=\sum_{n=0}^{\infty} a_n x^{n+1}=\sum_{n=1}^{\infty} a_{n-1}x^n$$

Hence, $y''=xy$ becomes $\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n=\sum_{n=1}^{\infty}a_{n-1}x^n$. We compare coefficients of x^n:

- $n=0$: $2\cdot 1 a_2=0$, so that $a_2=0$.
- $n\geqslant 1$: $(n+2)(n+1)a_{n+2}=a_{n-1}$
  Replacing n by $n-2$, this is equivalent to $n(n-1)a_n=a_{n-3}$ for $n\geqslant 3$.

In conclusion, $y(x)=\sum_{n=0}^{\infty}a_n x^n$ with $a_0=a$, $a_1=b$, $a_2=0$ as well as, for $n\geqslant 3$, $a_n=\frac{1}{n(n-1)}a_{n-3}$.

First few terms. In particular, $y=a\left(1+\frac{x^3}{2\cdot 3}+\frac{x^6}{(2\cdot 3)(5\cdot 6)}+\ldots\right)+b\left(x+\frac{x^4}{3\cdot 4}+\frac{x^7}{(3\cdot 4)(6\cdot 7)}+\ldots\right)$.

Advanced. The solution with $y(0)=\frac{1}{3^{2/3}\Gamma(2/3)}$ and $y'(0)=-\frac{1}{3^{1/3}\Gamma(1/3)}$ is known as the **Airy function** $\mathrm{Ai}(x)$. [A more natural property of $\mathrm{Ai}(x)$ is that it satisfies $y(x)\to 0$ as $x\to\infty$.]

Once we have a power series solution $y(x)$, a natural question is: for which x does the series converge?

Recall. A power series $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ has a **radius of convergence** R.
The series converges for all x with $|x - x_0| < R$ and it diverges for all x with $|x - x_0| > R$.

Definition 111. Consider the linear DE $y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \ldots + p_1(x) y' + p_0(x) y = f(x)$.
x_0 is called an **ordinary point** if the coefficients $p_j(x)$, as well as $f(x)$, are analytic at $x = x_0$.
Otherwise, x_0 is called a **singular point**.

Example 112. Determine the singular points of $(x+2)y'' - x^2 y + 3y = 0$.

Solution. Rewriting the DE as $y'' - \frac{x^2}{x+2} y + \frac{3}{x+2} y = 0$, we see that the only singular point is $x = -2$.

Example 113. Determine the singular points of $(x^2+1)y''' = \frac{y}{x-5}$.

Solution. Rewriting the DE as $y''' - \frac{1}{(x-5)(x^2+1)} y = 0$, we see that the singular points are $x = \pm i, 5$.

Theorem 114. Consider the linear DE $y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \ldots + p_1(x) y' + p_0(x) y = f(x)$.
Suppose that x_0 is an ordinary point and that R is the distance to the closest singular point.
Then any IVP specifying $y(x_0)$, $y'(x_0)$, ..., $y^{(n-1)}(x_0)$ has a power series solution $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ and that series has radius of convergence at least R.

In particular. The DE has a general solution consisting of n solutions $y(x)$ that are analytic at $x = x_0$.

Comment. Most textbooks only discuss the case of 2nd order DEs. For a discussion of the higher order case (in terms of first order systems!) see, for instance, Chapter 4.5 in *Ordinary Differential Equations* by N. Lebovitz.
The book is freely available at: `http://people.cs.uchicago.edu/~lebovitz/odes.html`

Example 115. Find a minimum value for the radius of convergence of a power series solution to $(x+2)y'' - x^2 y' + 3y = 0$ at $x = 3$.

Solution. As before, rewriting the DE as $y'' - \frac{x^2}{x+2} y' + \frac{3}{x+2} y = 0$, we see that the only singular point is $x = -2$.
Note that $x = 3$ is an ordinary point of the DE and that the distance to the singular point is $|3 - (-2)| = 5$.
Hence, the DE has power series solutions about $x = 3$ with radius of convergence at least 5.

Example 116. Find a minimum value for the radius of convergence of a power series solution to $(x^2+1)y''' = \frac{y}{x-5}$ at $x = 2$.

Solution. As before, rewriting the DE as $y''' - \frac{1}{(x-5)(x^2+1)} y = 0$, we see that the singular points are $x = \pm i, 5$.
Note that $x = 2$ is an ordinary point of the DE and that the distance to the nearest singular point is $|2 - i| = \sqrt{5}$ (the distances are $|2 - 5| = 3$, $|2 - i| = |2 - (-i)| = \sqrt{2^2 + 1^2} = \sqrt{5}$).
Hence, the DE has power series solutions about $x = 2$ with radius of convergence at least $\sqrt{5}$.

Example 117. (Airy equation, once more) Let $y(x)$ be the solution to the IVP $y'' = xy$, $y(0) = a$, $y'(0) = b$. Earlier, we determined the power series of $y(x)$. What is its radius of convergence?

Solution. $y'' = xy$ has no singular points. Hence, the radius of convergence is ∞. (In other words, the power series converges for all x.)

Armin Straub
straub@southalabama.edu