Review. If $y(x)$ is "nice" at $x = x_0$ (i.e. analytic around $x = x_0$), then

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad \text{with} \quad a_n = \frac{y^{(n)}(x_0)}{n!}.$$

In particular, at $x = 0$,

$$y(x) = \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!} x^n = y(0) + y'(0)x + \frac{1}{2}y''(0)x^2 + \frac{1}{6}y'''(0)x^3 + \ldots$$

Notation. When working with power series $\sum_{n=0}^{\infty} a_n x^n$, we sometimes write $O(x^n)$ to indicate that we omit terms that are multiples of x^n:

For instance. $e^x = 1 + x + \frac{1}{2}x^2 + O(x^3)$ or $\cos(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + O(x^6)$.

Example 108. Let $y(x)$ be the unique solution to the IVP $y' = x^2 + y^2$, $y(0) = 1$.
Determine the first several terms (up to x^4) in the power series of $y(x)$.

Solution. (successive differentiation) From the DE, $y'(0) = 0^2 + y(0)^2 = 1$.
Differentiating both sides of the DE, we obtain $y'' = 2x + 2yy'$. In particular, $y''(0) = 2$.
Continuing, $y''' = 2 + 2(y')^2 + 2yy''$ so that $y'''(0) = 2 + 2 + 2 \cdot 2 = 8$.
Likewise, $y^{(4)} = 6y'y'' + 2yy'''$ so that $y^{(4)}(0) = 12 + 16 = 28$.
Hence, $y(x) = y(0) + y'(0)x + \frac{1}{2}y''(0)x^2 + \frac{1}{6}y'''(0)x^3 + \frac{1}{24}y^{(4)}(0)x^4 + \ldots = 1 + x + x^2 + \frac{4}{3}x^3 + \frac{7}{6}x^4 + \ldots$

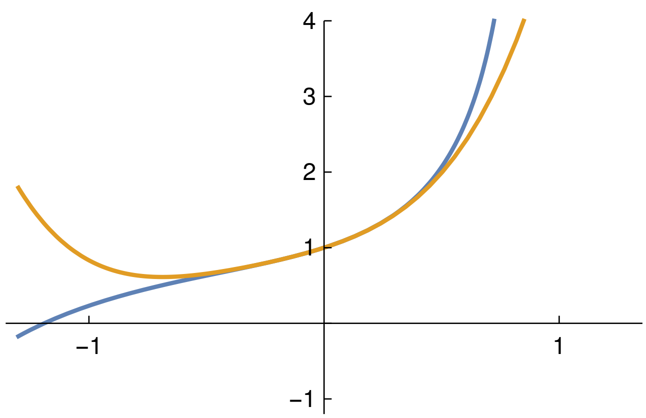
Comment. This approach requires the (symbolic) computation of intermediate derivatives. This is costly (even just the size of the simplified formulas is quickly increasing) and so the solution below is usually preferable for practical purposes. However, successive differentiation works well when working by hand.

Solution. (plug in power series) The powers series $y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots$ simplifies to $y = 1 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots$ because of the initial condition.
Therefore, $y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots$
To determine a_2, a_3, a_4, a_5, we need to expand $x^2 + y^2$ into a power series:

$y^2 = 1 + 2a_1x + (2a_2 + a_1^2)x^2 + (2a_3 + 2a_1a_2)x^3 + (2a_4 + 2a_1a_3 + a_2^2)x^4 + \ldots$ [we don't need the last term]

Equating coefficients of y' and $x^2 + y^2$, we find $a_1 = 1$, $2a_2 = 2a_1$, $3a_3 = 1 + 2a_2 + a_1^2$, $4a_4 = 2a_3 + 2a_1a_2$.
So $a_1 = 1$, $a_2 = 1$, $a_3 = \frac{4}{3}$, $a_4 = \frac{7}{6}$ and, hence, $y(x) = 1 + x + x^2 + \frac{4}{3}x^3 + \frac{7}{6}x^4 + \ldots$

Below is a plot of $y(x)$ (in blue) and our approximation:



Note how the approximation is very good close to 0 but does not provide us with a "global picture".

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Example 109. Let $y(x)$ be the unique solution to the IVP $y''=\cos(x+y)$, $y(0)=0$, $y'(0)=1$. Determine the first several terms (up to x^5) in the power series of $y(x)$.

Solution. (successive differentiation) From the DE, $y''(0)=\cos(0+y(0))=1$.
Differentiating both sides of the DE, we obtain $y'''=-\sin(x+y)(1+y')$.
In particular, $y'''(0)=-\sin(0+y(0))(1+y'(0))=0$.
Likewise, $y^{(4)}=-\cos(x+y)(1+y')^2-\sin(x+y)y''$ shows $y^{(4)}(0)=-1\cdot 2^2-0=-4$.
Continuing, $y^{(5)}=\sin(x+y)(1+y')^3-3\cos(x+y)(1+y')y''-\sin(x+y)y'''$ so that $y^{(5)}(0)=-6$.
Hence, $y(x)=x+\frac{1}{2}y''(0)x^2+\frac{1}{6}y'''(0)x^3+\frac{1}{24}y^{(4)}(0)x^4+\frac{1}{120}y^{(5)}(0)x^5+\ldots=x+\frac{1}{2}x^2-\frac{1}{6}x^4-\frac{1}{20}x^5+\ldots$

Solution. (plug in power series) The powers series $y=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\ldots$ simplifies to $y=x+a_2x^2+a_3x^3+a_4x^4+\ldots$ because of the initial conditions.
Therefore, $y'=1+2a_2x+3a_3x^2+4a_4x^3+\ldots$ and $y''=2a_2+6a_3x+12a_4x^2+20a_5x^3+\ldots$
To determine a_2,a_3,a_4,a_5, we need to expand $\cos(x+y)$ into a power series:

Recall that $\cos(x)=1-\frac{1}{2}x^2+\frac{1}{24}x^4+\ldots$

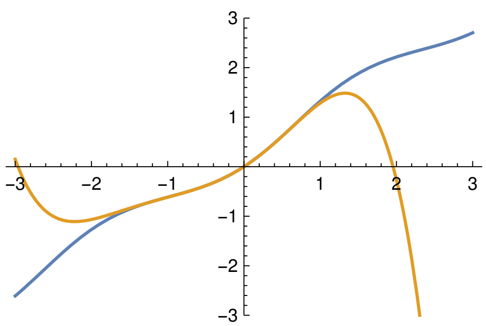
Hence, $\cos(x+y)=1-\frac{1}{2}(x+y)^2+\frac{1}{24}(x+y)^4+\ldots=1-\frac{1}{2}x^2-xy-\frac{1}{2}y^2+O(x^4)$.

Since $y^2=(x+a_2x^2+a_3x^3+\ldots)^2=x^2+2a_2x^3+O(x^4)$,

$\cos(x+y)=1-\frac{1}{2}x^2-x(x+a_2x^2)-\frac{1}{2}(x^2+2a_2x^3)+O(x^4)=1-2x^2-2a_2x^3+O(x^4)$.

Equating coefficients of y'' and $\cos(x+y)$, we find $2a_2=1$, $6a_3=0$, $12a_4=-2$, $20a_5=-2a_2$.
So $a_2=\frac{1}{2}$, $a_3=0$, $a_4=-\frac{1}{6}$, $a_5=-\frac{1}{20}$ and, hence, $y(x)=x+\frac{1}{2}x^2-\frac{1}{6}x^4-\frac{1}{20}x^5+\ldots$

Below is a plot of $y(x)$ (in blue) and our approximation:



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