Review. Lotka–Volterra predator–prey model

Power series solutions to DE

Given any DE, we can approximate analytic solutions by working with the first few terms of the power series.

Example 107. (Airy equation, part I) Let $y(x)$ be the unique solution to the IVP $y'' = xy$, $y(0) = a$, $y'(0) = b$. Determine the first several terms (up to x^6) in the power series of $y(x)$.

Solution. (successive differentiation) From the DE, $y''(0) = 0 \cdot y(0) = 0$.

Differentiating both sides of the DE, we obtain $y''' = y + xy'$ so that $y'''(0) = y(0) + 0 \cdot y'(0) = a$.

Likewise, $y^{(4)} = 2y' + xy''$ shows $y^{(4)}(0) = 2y'(0) = 2b$.

Continuing, $y^{(5)} = 3y'' + xy'''$ so that $y^{(5)}(0) = 3y''(0) = 0$.

Continuing, $y^{(6)} = 4y''' + xy^{(4)}$ so that $y^{(6)}(0) = 4y'''(0) = 4a$.

Hence, $y(x) = a + bx + \frac{1}{2}y''(0)x^2 + \frac{1}{6}y'''(0)x^3 + \frac{1}{24}y^{(4)}(0)x^4 + \frac{1}{120}y^{(5)}(0)x^5 + \frac{1}{720}y^{(6)}(0)x^6 + \ldots$
$= a + bx + \frac{a}{6}x^3 + \frac{b}{12}x^4 + \frac{a}{180}x^6 + \ldots$

Comment. Do you see the general pattern? We will revisit this example soon.

Solution. (plug in power series) The powers series $y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots$ becomes $y = a + bx + a_2x^2 + a_3x^3 + a_4x^4 + \ldots$ because of the initial conditions.

To determine a_2, a_3, a_4, a_5, a_6, we equate the coefficients of:

$$\begin{aligned} y'' &= 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + \ldots \\ xy &= ax + bx^2 + a_2x^3 + a_3x^4 + \ldots \end{aligned}$$

We find $2a_2 = 0$, $6a_3 = a$, $12a_4 = b$, $20a_5 = a_2$, $30a_6 = a_3$.

So $a_2 = 0$, $a_3 = \frac{a}{6}$, $a_4 = \frac{b}{12}$, $a_5 = \frac{a_2}{20} = 0$, $a_6 = \frac{a_3}{30} = \frac{a}{180}$. Hence, $y(x) = a + bx + \frac{a}{6}x^3 + \frac{b}{12}x^4 + \frac{a}{180}x^6 + \ldots$

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