Modeling

Example 96. Consider two brine tanks. Tank T_1 contains 24gal water containing 3lb salt, and tank T_2 contains 9gal pure water.

- T_1 is being filled with 54gal/min water containing 0.5lb/gal salt.
- 72gal/min well-mixed solution flows out of T_1 into T_2.
- 18gal/min well-mixed solution flows out of T_2 into T_1.
- Finally, 54gal/min well-mixed solution is leaving T_2.

We wish to understand how much salt is in the tanks after t minutes.

(a) Derive a system of differential equations.

(b) Determine the equilibrium points and classify their stability. What does this mean here?

(c) Solve the system to find explicit formulas for how much salt is in the tanks after t minutes.

Solution.

(a) Note that the amount of water in each tank is constant because the flows balance each other.
Let $y_i(t)$ denote the amount of salt (in lb) in tank T_i after time t (in min). In time interval $[t, t+\Delta t]$:
$\Delta y_1 \approx 54 \cdot \frac{1}{2} \cdot \Delta t - 72 \cdot \frac{y_1}{24} \cdot \Delta t + 18 \cdot \frac{y_2}{9} \cdot \Delta t$, so $y_1' = 27 - 3y_1 + 2y_2$. Also, $y_1(0) = 3$.
$\Delta y_2 \approx 72 \cdot \frac{y_1}{24} \cdot \Delta t - 72 \cdot \frac{y_2}{9} \cdot \Delta t$, so $y_2' = 3y_1 - 8y_2$. Also, $y_2(0) = 0$.
Using matrix notation and writing $\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, this is $\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{y} = \begin{bmatrix} -3 & 2 \\ 3 & -8 \end{bmatrix}\boldsymbol{y} + \begin{bmatrix} 27 \\ 0 \end{bmatrix}$, $\boldsymbol{y}(0) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$.

(b) Note that this system is autonomous! Otherwise, we could not pursue our stability analysis.
To find the equilibrium point (since the system is linear, there should be just one), we set $\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{y} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and solve $\begin{bmatrix} -3 & 2 \\ 3 & -8 \end{bmatrix}\boldsymbol{y} + \begin{bmatrix} 27 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. We find $\boldsymbol{y} = \begin{bmatrix} -3 & 2 \\ 3 & -8 \end{bmatrix}^{-1}\begin{bmatrix} -27 \\ 0 \end{bmatrix} = \begin{bmatrix} 12 \\ 4.5 \end{bmatrix}$.
The characteristic polynomial of $\begin{bmatrix} -3 & 2 \\ 3 & -8 \end{bmatrix}$ is $(-3-\lambda)(-8-\lambda) - 6 = \lambda^2 + 11\lambda + 18 = (\lambda+9)(\lambda+2)$.
Hence, the eigenvalues are $-9, -2$. Since they are both negative, the equilibrium point is a nodal sink and, in particular, asymptotically stable.
Having an equilibrium point at $(12, 4.5)$, means that, if the salt amounts are $y_1 = 12$, $y_2 = 4.5$, then they won't change over time (but will remain unchanged at these levels). The fact that it is asymptotically stable means that salt amounts close to these balanced levels will, over time, approach the equilibrium levels. (Because the system is linear, this is also true for levels that are not "close".)
We could have "seen" the equilibrium point!
Indeed, noticing that, for a constant (equilibrium) particular solution $\boldsymbol{y}$, each tank has to have a constant concentration of 0.5lb/gal of salt, we find directly $\boldsymbol{y} = 0.5\begin{bmatrix} 24 \\ 9 \end{bmatrix} = \begin{bmatrix} 12 \\ 4.5 \end{bmatrix}$.

(c) This is an IVP that we can solve (with some work). Do it! For instance, we can apply variation of constants. (Alternatively, leverage our particular solution from the previous part!) Skipping most work, we find:

- If $A = \begin{bmatrix} -3 & 2 \\ 3 & -8 \end{bmatrix}$, then $e^{At} = \frac{1}{7}\begin{bmatrix} e^{-9t} + 6e^{-2t} & -2e^{-9t} + 2e^{-2t} \\ -3e^{-9t} + 3e^{-2t} & 6e^{-9t} + 1e^{-2t} \end{bmatrix}$
- $\boldsymbol{y} = e^{At}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{At}\int_0^t e^{-As}\begin{bmatrix} 27 \\ 0 \end{bmatrix}\mathrm{d}s = \frac{1}{7}\begin{bmatrix} e^{-9t} + 6e^{-2t} \\ -3e^{-9t} + 3e^{-2t} \end{bmatrix} + \frac{3}{14}e^{At}\begin{bmatrix} 2e^{9t} + 54e^{2t} - 56 \\ -6e^{9t} + 27e^{2t} - 21 \end{bmatrix}$
  $= \begin{bmatrix} 12 - 9e^{-2t} \\ 4.5 - 4.5e^{-2t} \end{bmatrix}$

Armin Straub
straub@southalabama.edu