Systems of linear DEs: the inhomogeneous case

Recall that any linear DE can be transformed into a first-order system. Hence, any linear DE (or any system of linear DEs) can be written as

$$\boldsymbol{y}' = A(t)\,\boldsymbol{y} + \boldsymbol{f}(t).$$

Note. In general, A depends on t. In other words, the DE is allowed to have nonconstant coefficients. On the other hand, this is an autonomous DE (those for which we can analyze phase portraits) only if $A(t)$ and $\boldsymbol{f}(t)$ actually don't depend on t.

Review. We showed in Theorem 18 that $y' = a(t)y + f(t)$ has the particular solution

$$y_p(t) = y_h(t)\int \frac{f(t)}{y_h(t)}\mathrm{d}t,$$

where $y_h(t) = e^{\int a(t)\mathrm{d}t}$ is any solution to the homogeneous equation $y' = a(t)y$.

The same arguments with the same result apply to systems of linear equations!

Theorem 93. (variation of constants) $\boldsymbol{y}' = A(t)\,\boldsymbol{y} + \boldsymbol{f}(t)$ has the particular solution

$$\boldsymbol{y}_p(t) = \Phi(t)\int \Phi(t)^{-1}\boldsymbol{f}(t)\mathrm{d}t,$$

where $\Phi(t)$ is any fundamental matrix solution to $\boldsymbol{y}' = A(t)\,\boldsymbol{y}$.

Proof. We can find this formula in the same manner as we did in Theorem 18:
Since the general solution of the homogeneous equation $\boldsymbol{y}' = A(t)\,\boldsymbol{y}$ is $\boldsymbol{y}_h = \Phi(t)\boldsymbol{c}$, we are going to vary the constant $\boldsymbol{c}$ and look for a particular solution of the form $\boldsymbol{y}_p = \Phi(t)\boldsymbol{c}(t)$. Plugging into the DE, we get:

$\boldsymbol{y}_p' = \Phi'\boldsymbol{c} + \Phi\boldsymbol{c}' = A\Phi\boldsymbol{c} + \Phi\boldsymbol{c}' \overset{!}{=} A\boldsymbol{y}_p + \boldsymbol{f} = A\Phi\boldsymbol{c} + \boldsymbol{f}$

For the first equality, we used the matrix version of the usual product rule (which holds since differentiation is defined entry-wise). For the second equality, we used $\Phi' = A\Phi$.

Hence, $\boldsymbol{y}_p = \Phi(t)\boldsymbol{c}(t)$ is a particular solution if and only if $\Phi\boldsymbol{c}' = \boldsymbol{f}$.

The latter condition means $\boldsymbol{c}' = \Phi^{-1}\boldsymbol{f}$ so that $\boldsymbol{c} = \int \Phi(t)^{-1}\boldsymbol{f}(t)\mathrm{d}t$, which gives the claimed formula for $\boldsymbol{y}_p$. □

Example 94. Find a particular solution to $\boldsymbol{y}' = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}\boldsymbol{y} + \begin{bmatrix} 0 \\ -2e^{3t} \end{bmatrix}$.

Solution. First, we determine (do it!) a fundamental matrix solution for $\boldsymbol{y}' = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}\boldsymbol{y}$: $\Phi(x) = \begin{bmatrix} e^{-t} & 3e^{4t} \\ -e^{-t} & 2e^{4t} \end{bmatrix}$

Using $\det(\Phi(t)) = 5e^{3t}$, we find $\Phi(t)^{-1} = \frac{1}{5}\begin{bmatrix} 2e^{t} & -3e^{t} \\ e^{-4t} & e^{-4t} \end{bmatrix}$.

Hence, $\Phi(t)^{-1}\boldsymbol{f}(t) = \frac{2}{5}\begin{bmatrix} 3e^{4t} \\ -e^{-t} \end{bmatrix}$ and $\int \Phi(t)^{-1}\boldsymbol{f}(t)\mathrm{d}x = \frac{2}{5}\begin{bmatrix} 3/4e^{4t} \\ e^{-t} \end{bmatrix}$.

By variation of constants, $\boldsymbol{y}_p(t) = \Phi(t)\int \Phi(t)^{-1}\boldsymbol{f}(t)\mathrm{d}t = \begin{bmatrix} e^{-t} & 3e^{4t} \\ -e^{-t} & 2e^{4t} \end{bmatrix}\frac{2}{5}\begin{bmatrix} 3/4e^{4t} \\ e^{-t} \end{bmatrix} = \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix}e^{3t}$.

Comment. Note that the solution is of the form that we anticipate from the method of undetermined coefficients (which we only discussed in the case of a single DE but which works similarly for systems).

In the special case that $\Phi(t) = e^{At}$, some things become easier. For instance, $\Phi(t)^{-1} = e^{-At}$. In that case, we can explicitely write down solutions to IVPs:

- $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{c}$ has (unique) solution $\boldsymbol{y}(t) = e^{At}\boldsymbol{c}$.
- $\boldsymbol{y}' = A\boldsymbol{y} + \boldsymbol{f}(t)$, $\boldsymbol{y}(0) = \boldsymbol{c}$ has (unique) solution $\boldsymbol{y}(t) = e^{At}\boldsymbol{c} + e^{At}\int_0^t e^{-As}\boldsymbol{f}(s)\mathrm{d}s$.

Example 95. Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$.

(a) Determine e^{At}.

(b) Solve $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(c) Solve $\boldsymbol{y}' = A\boldsymbol{y} + \begin{bmatrix} 0 \\ 2e^t \end{bmatrix}$, $\boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Solution.

(a) By proceeding as in Example 74 (do it!), we find $e^{At} = \begin{bmatrix} 2e^{2t} - e^{3t} & -2e^{2t} + 2e^{3t} \\ e^{2t} - e^{3t} & -e^{2t} + 2e^{3t} \end{bmatrix}$.

(b) $\boldsymbol{y}(t) = e^{At}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2e^{2t} + 3e^{3t} \\ -e^{2t} + 3e^{3t} \end{bmatrix}$

(c) $\boldsymbol{y}(t) = e^{At}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + e^{At}\int_0^t e^{-As}\boldsymbol{f}(s)\mathrm{d}s$. We compute:

$\int_0^t e^{-As}\boldsymbol{f}(s)\mathrm{d}s = \int_0^t \begin{bmatrix} 2e^{-2s} - e^{-3s} & -2e^{-2s} + 2e^{-3s} \\ e^{-2s} - e^{-3s} & -e^{-2s} + 2e^{-3s} \end{bmatrix}\begin{bmatrix} 0 \\ 2e^s \end{bmatrix}\mathrm{d}s = \int_0^t \begin{bmatrix} -4e^{-s} + 4e^{-2s} \\ -2e^{-s} + 4e^{-2s} \end{bmatrix}\mathrm{d}s = \begin{bmatrix} 4e^{-t} - 2e^{-2t} - 2 \\ 2e^{-t} - 2e^{-2t} \end{bmatrix}$

Hence, $e^{At}\int_0^t e^{-As}\boldsymbol{f}(s)\mathrm{d}s = \begin{bmatrix} 2e^{2t} - e^{3t} & -2e^{2t} + 2e^{3t} \\ e^{2t} - e^{3t} & -e^{2t} + 2e^{3t} \end{bmatrix}\begin{bmatrix} 4e^{-t} - 2e^{-2t} - 2 \\ 2e^{-t} - 2e^{-2t} \end{bmatrix} = \begin{bmatrix} 2e^{t} - 4e^{2t} + 2e^{3t} \\ -2e^{2t} + 2e^{3t} \end{bmatrix}$.

Finally, $\boldsymbol{y}(t) = \begin{bmatrix} -2e^{2t} + 3e^{3t} \\ -e^{2t} + 3e^{3t} \end{bmatrix} + \begin{bmatrix} 2e^{t} - 4e^{2t} + 2e^{3t} \\ -2e^{2t} + 2e^{3t} \end{bmatrix} = \begin{bmatrix} 2e^{t} - 6e^{2t} + 5e^{3t} \\ -3e^{2t} + 5e^{3t} \end{bmatrix}$.

Sage. Here is how we can let Sage do these computations for us:

```
>>> s, t = var('s, t')
>>> A = matrix([[1,2],[-1,4]])
>>> y = exp(A*t)*vector([1,2]) + exp(A*t)*integrate(exp(-A*s)*vector([0,2*e^s]), s,0,t)
>>> y.simplify_full()
```

$$\left(5\, e^{(3\,t)} - 6\, e^{(2\,t)} + 2\, e^{t},\, 5\, e^{(3\,t)} - 3\, e^{(2\,t)}\right)$$

Comment. Can you see that the solution is of the form that we anticipate from the method of undetermined coefficients?
Indeed, $\boldsymbol{y}(t) = \boldsymbol{y}_p(t) + \boldsymbol{y}_h(t)$ where the simplest particular solution is $\boldsymbol{y}_p(t) = \begin{bmatrix} 2e^t \\ 0 \end{bmatrix}$.