Stability of autonomous linear differential equations

Example 87. (cont'd) Analyze the system $\frac{\mathrm{d}x}{\mathrm{d}t} = y - 5x + 3$, $\frac{\mathrm{d}y}{\mathrm{d}t} = 4x - 2y$.

In particular, determine the general solution as well as all equilibrium points and their stability.

Solution. Note that in Example 82, we looked at the corresponding homogeneous system and found that its general solution is $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-6t}$.

Note that we can write the present system in matrix form as $\begin{bmatrix} x \\ y \end{bmatrix}' = M \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ with $M = \begin{bmatrix} -5 & 1 \\ 4 & -2 \end{bmatrix}$.

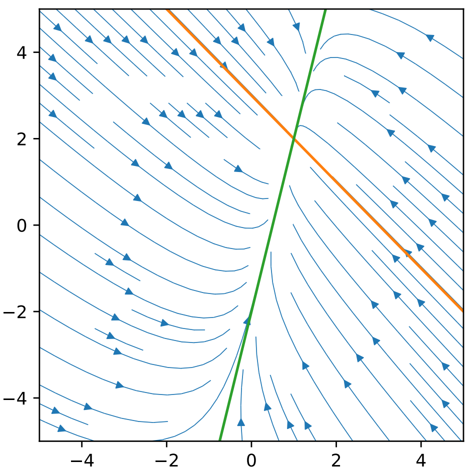
To find the equilibrium point, we solve $M \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} = 0$ to find $\begin{bmatrix} x \\ y \end{bmatrix} = -M^{-1} \begin{bmatrix} 3 \\ 0 \end{bmatrix} = -\frac{1}{6} \begin{bmatrix} -2 & -1 \\ -4 & -5 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

The fact that $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an equilibrium point means that $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is a particular solution!

(Make sure that you see that it has exactly the form we expect from the method of undetermined coefficients!)

Thus, the general solution must be $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-6t}$.

As a result, the phase portrait is going to look just as in Example 82 but shifted by $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$:



Because both eigenvalues (-1 and -6) are negative, $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an asymptotically stable equilibrium point. More precisely, it is what is called a **nodal source**.

As we have started to observe, the eigenvalues determine the stability of the equilibrium point in the case of an autonomous linear 2-dimensional systems. The following table gives an overview.

Important. Note that such a system must be of the form $\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} x \\ y \end{bmatrix} = M \begin{bmatrix} x \\ y \end{bmatrix} + \boldsymbol{c}$, where $\boldsymbol{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ is a constant vector. Because the system is autonomous, the matrix M and the inhomogeneous part $\boldsymbol{c}$ cannot depend on t.

(stability of autonomous linear 2-dimensional systems)

eigenvalues	**behaviour**	**stability**
real and both positive	nodal source	unstable
real and both negative	nodal sink	asymptotically stable
real and opposite signs	saddle	unstable
complex with positive real part	spiral source	unstable
complex with negative real part	spiral sink	asymptotically stable
purely imaginary	center point	stable but not asymptotically stable

Example 88. (spiral source, spiral sink, center point)

(a) Analyze the system $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

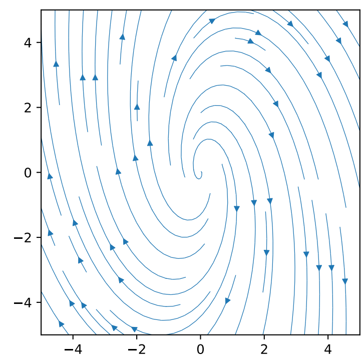
(b) Analyze the system $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = -\begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

(c) Analyze the system $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

Solution.

(a) The general solution, in real form, is:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1\begin{bmatrix} \cos(2t) \\ -2\sin(2t) \end{bmatrix}e^t + C_2\begin{bmatrix} \sin(2t) \\ 2\cos(2t) \end{bmatrix}e^t$$



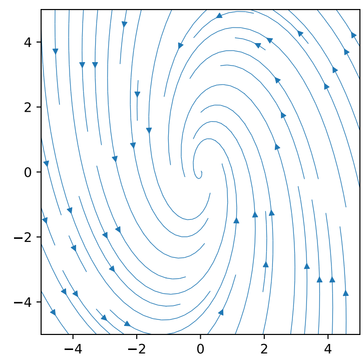
In this case, the origin is a spiral source which is an unstable equilibrium (note that it follows from $e^t \to \infty$ as $t \to \infty$ that all solutions "flow away" from the origin because they have increasing amplitude).

Review. $\begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$ parametrizes the unit circle.

Similarly, $\begin{bmatrix} \cos(t) \\ 2\sin(t) \end{bmatrix}$ parametrizes an ellipse.

(b) The general solution, in real form, is:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1\begin{bmatrix} \cos(2t) \\ -2\sin(2t) \end{bmatrix}e^{-t} + C_2\begin{bmatrix} \sin(2t) \\ 2\cos(2t) \end{bmatrix}e^{-t}$$

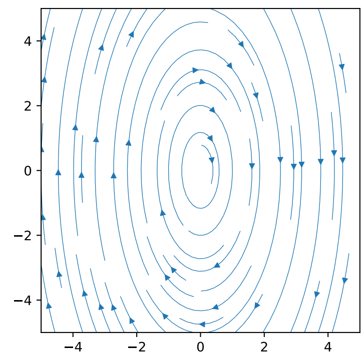


In this case, the origin is a spiral sink which is an asymptotically stable equilibrium (note that it follows from $e^{-t} \to 0$ as $t \to \infty$ that all solutions "flow into" the origin because their amplitude goes to zero).

Comment. Note that $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ solves the first system if and only if $\begin{bmatrix} x(-t) \\ y(-t) \end{bmatrix}$ is a solution to the second. Consequently, the phase portraits look alike but all arrows are reversed.

(c) The general solution, in real form, is:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1\begin{bmatrix} \cos(2t) \\ -2\sin(2t) \end{bmatrix} + C_2\begin{bmatrix} \sin(2t) \\ 2\cos(2t) \end{bmatrix}$$



In this case, the origin is a center point which is a stable equilibrium (note that the solutions are periodic with period π and therefore loop around the origin; with each trajectory a perfect ellipse).

Armin Straub
straub@southalabama.edu