Phase portraits and phase plane analysis

Our goal is to visualize the solutions to systems of equations. This works particularly well in the case of systems of two differential equations.

A system of two differential equations that can be written as

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= f(x,y)\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= g(x,y)\end{aligned}$$

is called **autonomous** because it doesn't depend on the independent variable t.

Comment. Can you show that if $x(t)$ and $y(t)$ are a pair of solutions, then so is the pair $x(t+t_0)$ and $y(t+t_0)$?

We can visualize solutions to such a system by plotting the points $(x(t), y(t))$ for increasing values of t so that we get a curve (and we can attach an arrow to indicate the direction we're flowing along that curve). Each such curve is called the **trajectory** of a solution.

Even better, we can do such a **phase portrait** without solving to get a formula for $(x(t), y(t))$! That's because we can combine the two equations to get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{g(x,y)}{f(x,y)},$$

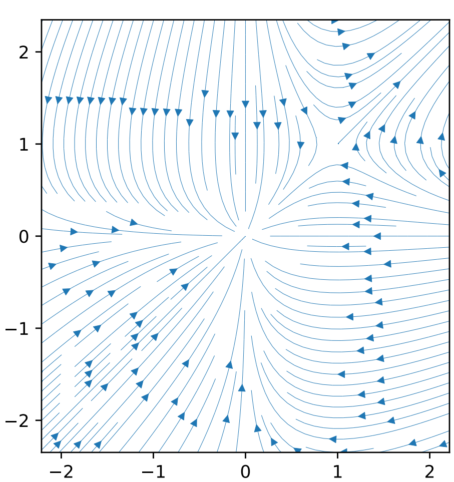
which allows us to make a slope field! If a trajectory passes through a point (x, y), then we know that the slope at that point must be $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{g(x,y)}{f(x,y)}$.

This allows us to sketch trajectories. However, it does not tell us everything about the corresponding solution $(x(t), y(t))$ because we don't know at which times t the solution passes through the points on the curve.

However, we can visualize the speed with which a solution passes through the trajectory by attaching to a point (x, y) not only the slope $\frac{g(x,y)}{f(x,y)}$ but the vector $\begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$. That vector has the same direction as the slope but it also tells us in which direction we are moving and how fast (by its magnitude).

Example 80. Sketch some trajectories for the system $\frac{\mathrm{d}x}{\mathrm{d}t} = x \cdot (y-1)$, $\frac{\mathrm{d}y}{\mathrm{d}t} = y \cdot (x-1)$.

Solution.



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Comment. In this example, we can solve the slope-field equation $\frac{dy}{dx} = \frac{y(x-1)}{x(y-1)}$ using separation of variables.

Do it! We end up with the implicit solutions $y - \ln|y| = x - \ln|x| + C$.

If we plot these curves for various values of C, we get trajectories in the plot above. However, note that none of this solving is needed for plotting by itself.

Sage. We can make Sage create such phase portraits for us!

```
>>> x,y = var('x y')
>>> plot = streamline_plot((x*(y-1),y*(x-1)), (x,-3,3), (y,-3,3))
```

Equilibrium solutions

(x_0, y_0) is an **equilibrium point** of the system $\frac{dx}{dt} = f(x,y)$, $\frac{dy}{dt} = g(x,y)$ if

$$f(x_0, y_0) = 0 \quad \text{and} \quad g(x_0, y_0) = 0.$$

In that case, we have the **equilibrium solution** $x(t) = x_0$, $y(t) = y_0$.

Comment. Equilibrium points are also called **critical points** (or stationary points or rest points).

In a phase portrait, the equilibrium solutions are just a single point.

Recall that every other solution $(x(t), y(t))$ corresponds to a curve (parametrized by t), called the **trajectory** of the solution (and we can adorn it with an arrow that indicates the direction of the "flow" of the solution).

We can learn a lot from how solutions behave near equilibrium points.

An equilibrium point is called:

- **stable** if all nearby solutions remain close to the equilibrium point;
- **asymptotically stable** if all nearby solutions remain close and "flow into" the equilibrium;
- **unstable** if it is not stable (some nearby solutions "flow away" from the equilibrium).

Comment. Note that asymptotically stable is a stronger condition than stable. A typical example of a stable, but not asymptotically stable, equilibrium point is one where nearby solutions loop around the equilibrium point without coming closer to it.

Advanced comment. For asymptotically stable, we kept the condition that nearby solutions remain close because there are "weird" instances where trajectories come arbitrarily close to the equilibrium, then "flow away" but eventually "flow into" (this would constitute an unstable equilibrium point).

Example 81. (cont'd) Consider again the system $\frac{dx}{dt} = x \cdot (y-1)$, $\frac{dy}{dt} = y \cdot (x-1)$.

(a) Determine the equilibrium points.

(b) Using our earlier phase portrait, classify the stability of each equilibrium point.

Solution.

(a) We solve $x(y-1) = 0$ (that is, $x = 0$ or $y = 1$) and $y(x-1) = 0$ (that is, $x = 1$ or $y = 0$).
We conclude that the equilibrium points are $(0,0)$ and $(1,1)$.

(b) $(0,0)$ is asymptotically stable (because all nearby solutions "flow into" $(0,0)$).
$(1,1)$ is unstable (because some nearby solutions "flow away" from $(1,1)$).
Comment. We will soon learn how to determine stability without the need for a plot.
Comment. If you look carefully at the phase portrait near $(1,1)$, you can see that certain solutions get attracted at first to $(1,1)$ and then "flow away" at the last moment. This suggests that there is a single trajectory which actually "flows into" $(1,1)$. This constellation is typical and is called a **saddle point**.

Phase portraits of linear differential equations

Example 82. Consider the system $\frac{\mathrm{d}x}{\mathrm{d}t} = y - 5x$, $\frac{\mathrm{d}y}{\mathrm{d}t} = 4x - 2y$.

(a) Determine the general solution.

(b) Make a phase portrait. Can you connect it with the general solution?

(c) Determine all equilibrium points and their stability.

Solution.

(a) Note that we can write this is in matrix form as $\begin{bmatrix} x \\ y \end{bmatrix}' = M \begin{bmatrix} x \\ y \end{bmatrix}$ with $M = \begin{bmatrix} -5 & 1 \\ 4 & -2 \end{bmatrix}$.

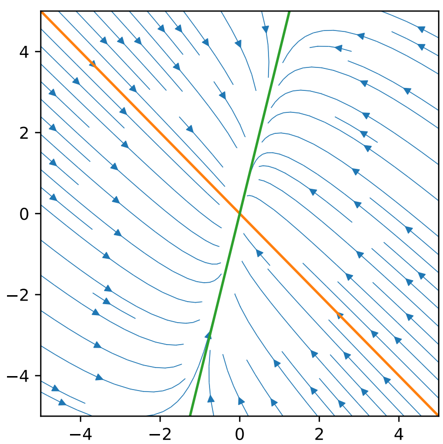
M has -1-eigenvector $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ as well as -6-eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

The general solution is $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-6t}$.

(b) We can have Sage make such a plot for us:

```
>>> x,y = var('x y')
    plot = streamline_plot((-5*x+y,4*x-2*y), (x,-4,4), (y,-4,4))
```

Question. In our plot, we also highlighted two lines through the origin. Can you explain their significance?



Explanation. The two lines correspond to the special solutions $C_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{-t}$ and $C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-6t}$. In each case, the trajectories consist of points that are multiples of the vectors $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, respectively.

Note that each such solution starts at a point on one of the lines and then "flows" into the origin. (Because e^{-t} and e^{-6t} approach zero for large t.)

Question. Consider a point like $(4, 4)$. Can you explain why the trajectory through that point doesn't go somewhat straight to $(0, 0)$ but rather flows nearly parallel the orange line towards the green line?

Explanation. A solution through $(4, 4)$ is of the form $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-6t}$ (like any other solution). Note that, if we increase t, then e^{-6t} becomes small much faster than e^{-t}.

As a consequence, we quickly get $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \approx C_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{-t}$, where the right-hand side is on the green line.

(c) The only equilibrium point is $(0, 0)$ and it is asymptotically stable.

We can see this from the phase portrait but we can also determine it from the DE and our solution: first, solving $y - 5x = 0$ and $4x - 2y = 0$ we only get the unique solution $x = 0, y = 0$, which means that only $(0, 0)$ is an equilibrium point. On the other hand, the general solution shows that every solution approaches $(0, 0)$ as $t \to \infty$ because both e^{-t} and e^{-6t} approach 0.

In general. This is typical: if both eigenvalues are negative, then the equilibrium is asymptotically stable. It at least one eigenvalue is positive, then the equilibrium is unstable.

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