Example 75. (review) Let $M = \begin{bmatrix} -1 & 6 \\ -1 & 4 \end{bmatrix}$.

(a) Compute e^{Mx}.

(b) Solve the initial value problem $\boldsymbol{y}' = M\boldsymbol{y}$ with $\boldsymbol{y}(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

Solution.

(a) We determine the eigenvectors of M. The characteristic polynomial is:
$\det(M - \lambda I) = \det\left(\begin{bmatrix} -1-\lambda & 6 \\ -1 & 4-\lambda \end{bmatrix}\right) = (-1-\lambda)(4-\lambda) + 6 = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2)$
Hence, the eigenvalues are $\lambda = 1$ and $\lambda = 2$.

- $\lambda = 1$: Solving $\begin{bmatrix} -2 & 6 \\ -1 & 3 \end{bmatrix}\boldsymbol{v} = \boldsymbol{0}$, we find that $\boldsymbol{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 1$.
- $\lambda = 2$: Solving $\begin{bmatrix} -3 & 6 \\ -1 & 2 \end{bmatrix}\boldsymbol{v} = \boldsymbol{0}$, we find that $\boldsymbol{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 2$.

Hence, a fundamental matrix solution is $\Phi = \begin{bmatrix} 3e^x & 2e^{2x} \\ e^x & e^{2x} \end{bmatrix}$.

Note that $\Phi(0) = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, so that $\Phi(0)^{-1} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$. It follows that

$$e^{Mx} = \Phi(x)\Phi(0)^{-1} = \begin{bmatrix} 3e^x & 2e^{2x} \\ e^x & e^{2x} \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 3e^x - 2e^{2x} & -6e^x + 6e^{2x} \\ e^x - e^{2x} & -2e^x + 3e^{2x} \end{bmatrix}.$$

(b) The solution to the IVP is $\boldsymbol{y}(x) = e^{Mx}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 3e^x - 2e^{2x} \\ e^x - e^{2x} \end{bmatrix}$ (twice the first column of e^{Mx}).

Another perspective on the matrix exponential

(exponential function) e^x is the unique solution to $y' = y$, $y(0) = 1$.
From here, it follows that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$.

The latter is the Taylor series for e^x at $x = 0$ that we have seen in Calculus II.
Important note. We can actually construct this infinite sum directly from $y' = y$ and $y(0) = 1$.
Indeed, observe how each term, when differentiated, produces the term before it. For instance, $\frac{\mathrm{d}}{\mathrm{d}x}\frac{x^3}{3!} = \frac{x^2}{2!}$.

Review. We defined the **matrix exponential** e^{Mx} as the unique matrix solution to the IVP

$$\boldsymbol{y}' = M\boldsymbol{y}, \quad \boldsymbol{y}(0) = I.$$

Below, we observe that we can also make sense of the matrix exponential e^{Mx} as a power series.

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Theorem 76. Let M be $n\times n$. Then the **matrix exponential** satisfies

$$e^M = I + M + \frac{1}{2!}M^2 + \frac{1}{3!}M^3 + \ldots$$

Proof. Define $\Phi(x) = I + Mx + \frac{1}{2!}M^2x^2 + \frac{1}{3!}M^3x^3 + \ldots$

$$\begin{aligned}\Phi'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left[I + Mx + \frac{1}{2!}M^2x^2 + \frac{1}{3!}M^3x^3 + \ldots\right]\\ &= 0 + M + M^2x + \frac{1}{2!}M^3x^2 + \ldots = M\Phi(x).\end{aligned}$$

Clearly, $\Phi(0) = I$. Therefore, $\Phi(x)$ is the fundamental matrix solution to $\boldsymbol{y}' = M\boldsymbol{y}$, $\boldsymbol{y}(0) = I$.
But that's precisely how we defined e^{Mx} earlier. It follows that $\Phi(x) = e^{Mx}$. Now set $x = 1$. □

Example 77. If $A = \begin{bmatrix} 2 & 0\\ 0 & 5\end{bmatrix}$, then $A^{100} = \begin{bmatrix} 2^{100} & 0\\ 0 & 5^{100}\end{bmatrix}$.

Example 78. If $A = \begin{bmatrix} 2 & 0\\ 0 & 5\end{bmatrix}$, then $e^A = \begin{bmatrix} 1 & 0\\ 0 & 1\end{bmatrix} + \begin{bmatrix} 2 & 0\\ 0 & 5\end{bmatrix} + \frac{1}{2!}\begin{bmatrix} 2^2 & 0\\ 0 & 5^2\end{bmatrix} + \cdots = \begin{bmatrix} e^2 & 0\\ 0 & e^5\end{bmatrix}$.

Clearly, this works to obtain e^D for every diagonal matrix D.

In particular, for $Ax = \begin{bmatrix} 2x & 0\\ 0 & 5x\end{bmatrix}$, $e^{Ax} = \begin{bmatrix} 1 & 0\\ 0 & 1\end{bmatrix} + \begin{bmatrix} 2x & 0\\ 0 & 5x\end{bmatrix} + \frac{1}{2!}\begin{bmatrix} (2x)^2 & 0\\ 0 & (5x)^2\end{bmatrix} + \cdots = \begin{bmatrix} e^{2x} & 0\\ 0 & e^{5x}\end{bmatrix}$.

Extra: The case of repeated eigenvalues with too few eigenvectors

Review. To construct a fundamental matrix solution $\Phi(x)$ to $\boldsymbol{y}' = M\boldsymbol{y}$, we compute eigenvectors:
Given a λ-eigenvector $\boldsymbol{v}$, we have the corresponding solution $\boldsymbol{y}(x) = \boldsymbol{v}e^{\lambda x}$.
If there are enough eigenvectors, we can collect these as columns to obtain $\Phi(x)$.
The next example illustrates how to proceed if there are not enough eigenvectors.

In that case, instead of looking only for solutions of the type $\boldsymbol{y}(x) = \boldsymbol{v}e^{\lambda x}$, we also need to look for solutions of the type $\boldsymbol{y}(x) = (\boldsymbol{v}x + \boldsymbol{w})e^{\lambda x}$. This can only happen if an eigenvalue is a repeated root of the characteristic polynomial.

Example 79. Let $M = \begin{bmatrix} 8 & 4\\ -1 & 4\end{bmatrix}$.

(a) Determine the general solution to $\boldsymbol{y}' = M\boldsymbol{y}$.

(b) Determine a fundamental matrix solution to $\boldsymbol{y}' = M\boldsymbol{y}$.

(c) Compute e^{Mx}.

(d) Solve the initial value problem $\boldsymbol{y}' = M\boldsymbol{y}$ with $\boldsymbol{y}(0) = \begin{bmatrix} 1\\ 0\end{bmatrix}$.

Solution.

(a) We determine the eigenvectors of M. The characteristic polynomial is:

$\det(M-\lambda I)=\det\left(\begin{bmatrix} 8-\lambda & 4 \\ -1 & 4-\lambda \end{bmatrix}\right)=(8-\lambda)(4-\lambda)+4=\lambda^2-12\lambda+36=(\lambda-6)(\lambda-6)$

Hence, the eigenvalues are $\lambda=6,6$ (meaning that 6 has multiplicity 2).

- To find eigenvectors $\boldsymbol{v}$ for $\lambda=6$, we need to solve $\begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}\boldsymbol{v}=\boldsymbol{0}$.

  Hence, $\boldsymbol{v}=\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda=6$. There is no independent second eigenvector.

- We therefore search for a solution of the form $\boldsymbol{y}(x)=(\boldsymbol{v}x+\boldsymbol{w})e^{\lambda x}$ with $\lambda=6$.

  $\boldsymbol{y}'(x)=(\lambda\boldsymbol{v}x+\lambda\boldsymbol{w}+\boldsymbol{v})e^{\lambda x}\overset{!}{=}M\boldsymbol{y}=(M\boldsymbol{v}x+M\boldsymbol{w})e^{\lambda x}$

  Equating coefficients of x, we need $\lambda\boldsymbol{v}=M\boldsymbol{v}$ and $\lambda\boldsymbol{w}+\boldsymbol{v}=M\boldsymbol{w}$.

  Hence, $\boldsymbol{v}$ must be an eigenvector (which we already computed); we choose $\boldsymbol{v}=\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

  [Note that any multiple of $\boldsymbol{y}(x)$ will be another solution, so it doesn't matter which multiple of $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ we choose.]

  $\lambda\boldsymbol{w}+\boldsymbol{v}=M\boldsymbol{w}$ or $(M-\lambda)\boldsymbol{w}=\boldsymbol{v}$ then becomes $\begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}\boldsymbol{w}=\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

  One solution is $\boldsymbol{w}=\begin{bmatrix} -1 \\ 0 \end{bmatrix}$. [We only need one.]

Hence, the general solution is $C_1\begin{bmatrix} -2 \\ 1 \end{bmatrix}e^{6x}+C_2\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}x+\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right)e^{6x}$.

(b) The corresponding fundamental matrix solution is $\Phi=\begin{bmatrix} -2e^{6x} & -(2x+1)e^{6x} \\ e^{6x} & x\,e^{6x} \end{bmatrix}$.

(c) Note that $\Phi(0)=\begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix}$, so that $\Phi(0)^{-1}=\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$. It follows that

$$e^{Mx}=\Phi(x)\Phi(0)^{-1}=\begin{bmatrix} -2e^{6x} & -(2x+1)e^{6x} \\ e^{6x} & x\,e^{6x} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}=\begin{bmatrix} (2x+1)e^{6x} & 4x\,e^{6x} \\ -x\,e^{6x} & -(2x-1)\,e^{6x} \end{bmatrix}.$$

(d) The solution to the IVP is $\boldsymbol{y}(x)=e^{Mx}\begin{bmatrix} 1 \\ 0 \end{bmatrix}=\begin{bmatrix} (2x+1)e^{6x} & 4x\,e^{6x} \\ -x\,e^{6x} & -(2x-1)\,e^{6x} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}=\begin{bmatrix} (2x+1)e^{6x} \\ -x\,e^{6x} \end{bmatrix}$.