(systems of REs) The unique solution to $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$, $\boldsymbol{a}_0 = \boldsymbol{c}$ is $\boldsymbol{a}_n = M^n\boldsymbol{c}$.

- Here, M^n is the fundamental matrix solution to $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$, $\boldsymbol{a}_0 = I$ (with I the identity matrix).
- If Φ_n is any fundamental matrix solution to $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$, then $M^n = \Phi_n\Phi_0^{-1}$.
- To construct a fundamental matrix solution Φ_n, we compute eigenvectors:
  Given a λ-eigenvector $\boldsymbol{v}$, we have the corresponding solution $\boldsymbol{a}_n = \boldsymbol{v}\lambda^n$.
  If there are enough eigenvectors, we can collect these as columns to obtain Φ_n.

Why? Since Φ_n is a fundamental matrix solution, we have $\Phi_{n+1} = M\Phi_n$ and, thus, $\Phi_n = M^n\Phi_0$.
It follows that $M^n = \Phi_n\Phi_0^{-1}$.

Example 63. (review) Write the (second-order) RE $a_{n+2} = a_{n+1} + 2a_n$, with $a_0 = 0$, $a_1 = 1$, as a system of (first-order) recurrences.

Solution. If $\boldsymbol{a}_n = \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix}$, then $\boldsymbol{a}_{n+1} = \begin{bmatrix} a_{n+1} \\ a_{n+2} \end{bmatrix} = \begin{bmatrix} a_{n+1} \\ a_{n+1} + 2a_n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}\boldsymbol{a}_n$ with $\boldsymbol{a}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Example 64. Let $M = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$.

(a) Determine the general solution to $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$.

(b) Determine a fundamental matrix solution to $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$.

(c) Compute M^n.

(d) Solve $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$, $\boldsymbol{a}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Solution.

(a) Recall that each λ-eigenvector $\boldsymbol{v}$ of M provides us with a solution: namely, $\boldsymbol{a}_n = \boldsymbol{v}\lambda^n$.
The characteristic polynomial is: $\det(A - \lambda I) = \det\left(\begin{bmatrix} -\lambda & 1 \\ 2 & 1-\lambda \end{bmatrix}\right) = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)$.
Hence, the eigenvalues are $\lambda = 2$ and $\lambda = -1$.

- $\lambda = 2$: Solving $\begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix}\boldsymbol{v} = \boldsymbol{0}$, we find that $\boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector for $\lambda = 2$.
- $\lambda = -1$: Solving $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\boldsymbol{v} = \boldsymbol{0}$, we find that $\boldsymbol{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = -1$.

Hence, the general solution is $C_1\begin{bmatrix} 1 \\ 2 \end{bmatrix}2^n + C_2\begin{bmatrix} -1 \\ 1 \end{bmatrix}(-1)^n$.

(b) Note that $C_1\begin{bmatrix} 1 \\ 2 \end{bmatrix}2^n + C_2\begin{bmatrix} -1 \\ 1 \end{bmatrix}(-1)^n = \begin{bmatrix} 2^n & -(-1)^n \\ 2\cdot 2^n & (-1)^n \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$.
Hence, a fundamental matrix solution is $\Phi_n = \begin{bmatrix} 2^n & -(-1)^n \\ 2\cdot 2^n & (-1)^n \end{bmatrix}$.
Comment. Other choices are possible and natural. For instance, the order of the two columns is based on our choice of starting with $\lambda = 2$. Also, the columns can be scaled by any constant (for instance, using $-\boldsymbol{v}$ instead of $\boldsymbol{v}$ for $\lambda = -1$ above, we end up with the same Φ_n but with the second column scaled by -1).
In general, if Φ_n is a fundamental matrix solution, then so is $\Phi_n C$ where C is an invertible 2×2 matrix.

(c) We compute $M^n = \Phi_n\Phi_0^{-1}$ using $\Phi_n = \begin{bmatrix} 2^n & -(-1)^n \\ 2\cdot 2^n & (-1)^n \end{bmatrix}$. Since $\Phi_0^{-1} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$, we have

$$M^n = \Phi_n\Phi_0^{-1} = \begin{bmatrix} 2^n & -(-1)^n \\ 2\cdot 2^n & (-1)^n \end{bmatrix}\frac{1}{3}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2^n + 2(-1)^n & 2^n - (-1)^n \\ 2\cdot 2^n - 2(-1)^n & 2\cdot 2^n + (-1)^n \end{bmatrix}.$$

(d) $\boldsymbol{a}_n = M^n\boldsymbol{a}_0 = \frac{1}{3}\begin{bmatrix} 2^n + 2(-1)^n & 2^n - (-1)^n \\ 2\cdot 2^n - 2(-1)^n & 2\cdot 2^n + (-1)^n \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2^n - (-1)^n \\ 2\cdot 2^n + (-1)^n \end{bmatrix}$

Alternative solution of the first part. We saw in Example 63 that this system can be obtained from $a_{n+2} = a_{n+1} + 2a_n$ if we set $\boldsymbol{a} = \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix}$. In Example 55, we found that this RE has solutions $a_n = 2^n$ and $a_n = (-1)^n$. Correspondingly, $\boldsymbol{a}_{n+1} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \boldsymbol{a}_n$ has solutions $\boldsymbol{a}_n = \begin{bmatrix} 2^n \\ 2^{n+1} \end{bmatrix}$ and $\boldsymbol{a}_n = \begin{bmatrix} (-1)^n \\ (-1)^{n+1} \end{bmatrix}$.

These combine to the general solution $C_1 \begin{bmatrix} 2^n \\ 2^{n+1} \end{bmatrix} + C_2 \begin{bmatrix} (-1)^n \\ (-1)^{n+1} \end{bmatrix}$ (equivalent to our solution above).

Alternative for last part. Solve the RE from Example 63 to find $a_n = \frac{1}{3}(2^n - (-1)^n)$. The above is $\boldsymbol{a}_n = \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix}$.

Sage. Once we are comfortable with these computations, we can let Sage do them for us.

```
>>> M = matrix([[0,1],[2,1]])
>>> M^2
```

$$\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$$

Verify that this matrix matches what our formula for M^n produces for $n = 2$. In order to reproduce the general formula for M^n, we need to first define n as a symbolic variable:

```
>>> n = var('n')
>>> M^n
```

$$\begin{pmatrix} \frac{1}{3} \cdot 2^n + \frac{2}{3} (-1)^n & \frac{1}{3} \cdot 2^n - \frac{1}{3} (-1)^n \\ \frac{2}{3} \cdot 2^n - \frac{2}{3} (-1)^n & \frac{2}{3} \cdot 2^n + \frac{1}{3} (-1)^n \end{pmatrix}$$

Note that this indeed matches our earlier formula. Can you see how we can read off the eigenvalues and eigenvectors of M from this formula for M^n? Of course, Sage can readily compute these for us directly using, for instance, `M.eigenvectors_right()`. Try it! Can you interpret the output?

Preview: The corresponding system of differential equations

Example 65. Write the (second-order) initial value problem $y'' = y' + 2y$, $y(0) = 0$, $y'(0) = 1$ as a first-order system.

Solution. If $\boldsymbol{y} = \begin{bmatrix} y \\ y' \end{bmatrix}$, then $\boldsymbol{y}' = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} y' \\ y' + 2y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \boldsymbol{y}$ with $\boldsymbol{y}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Homework. Solve this IVP to find $y(x) = \frac{1}{3}(e^{2x} - e^{-x})$. Then compare with the next example.

Example 66. Let $M = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$.

(a) Determine the general solution to $\boldsymbol{y}' = M\boldsymbol{y}$.

(b) Determine a fundamental matrix solution to $\boldsymbol{y}' = M\boldsymbol{y}$.

(c) Solve $\boldsymbol{y}' = M\boldsymbol{y}$, $\boldsymbol{y}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Solution. In Example 63, we only need to replace 2^n by e^{2x} (root 2) and $(-1)^n$ by e^{-x} (root -1)!

(a) The general solution is $C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2x} + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-x}$.

(b) A fundamental matrix solution is $\Phi(x) = \begin{bmatrix} e^{2x} & -e^{-x} \\ 2 \cdot e^{2x} & e^{-x} \end{bmatrix}$.

(c) $\boldsymbol{y}(x) = \frac{1}{3} \begin{bmatrix} e^{2x} - e^{-x} \\ 2 \cdot e^{2x} + e^{-x} \end{bmatrix}$

Preview. The special fundamental matrix M^n will be replaced by e^{Mx}, the **matrix exponential**.

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Example 67. (homework)

(a) Write the recurrence $a_{n+3} - 4a_{n+2} + a_{n+1} + 6a_n = 0$ as a system $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$ of (first-order) recurrences.

(b) Determine a fundamental matrix solution to $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$.

(c) Compute M^n.

Solution.

(a) If $\boldsymbol{a}_n = \begin{bmatrix} a_n \\ a_{n+1} \\ a_{n+2} \end{bmatrix}$, then the RE becomes $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$ with $M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -1 & 4 \end{bmatrix}$.

(b) Because we started with a single (third-order) equation, we can avoid computing eigenvectors and eigenvalues (indeed, we will find these as a byproduct).

By factoring the characteristic equation $N^3 - 4N^2 + N + 6 = (N-3)(N-2)(N+1)$, we find that the characteristic roots are $3, 2, -1$ (these are also precisely the eigenvalues of M).

Hence, $a_n = C_1 \cdot 3^n + C_2 \cdot 2^n + C_3 \cdot (-1)^n$ is the general solution to the initial RE.

Correspondingly, a fundamental matrix solution of the system is $\Phi_n = \begin{bmatrix} 3^n & 2^n & (-1)^n \\ 3 \cdot 3^n & 2 \cdot 2^n & -(-1)^n \\ 9 \cdot 3^n & 4 \cdot 2^n & (-1)^n \end{bmatrix}$.

Note. This tells us that $\begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$ is a 3-eigenvector, $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ a 2-eigenvector, and $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ a -1-eigenvector of M.

(c) Since $\Phi_{n+1} = M\Phi_n$, we have $\Phi_n = M^n \Phi_0$ so that $M^n = \Phi_n \Phi_0^{-1}$. This allows us to compute that:

$$M^n = \frac{1}{12}\begin{bmatrix} -6 \cdot 3^n + 12 \cdot 2^n + 6(-1)^n & -3 \cdot 3^n + 8 \cdot 2^n - 5(-1)^n & 3 \cdot 3^n - 4 \cdot 2^n + (-1)^n \\ -18 \cdot 3^n + 24 \cdot 2^n - 6(-1)^n & \ldots & \ldots \\ -54 \cdot 3^n + 48 \cdot 2^n + 6(-1)^n & \ldots & \ldots \end{bmatrix}$$

Example 68. (homework) If $M = \begin{bmatrix} 3 & & & \\ & -2 & & \\ & & 5 & \\ & & & 1 \end{bmatrix}$, what is M^n?

Comment. Entries that are not printed are meant to be zero (to make the structure of the 4×4 matrix more visibly transparent).

Solution. $M^n = \begin{bmatrix} 3^n & & & \\ & (-2)^n & & \\ & & 5^n & \\ & & & 1 \end{bmatrix}$

If this isn't clear to you, multiply out M^2. What happens?