Example 59. (review) Consider the sequence a_n defined by $a_{n+2} = 4a_n - 3a_{n+1}$ and $a_0 = 1$, $a_1 = 2$. Determine $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

Solution. The recursion can be written as $p(N)a_n = 0$ where $p(N) = N^2 + 3N - 4$ has roots $1, -4$. Hence, the general solution is $a_n = C_1 + C_2 \cdot (-4)^n$. We can see that both roots have to be involved in the solution (in other words, $C_1 \neq 0$ and $C_2 \neq 0$) because $a_n = C_1$ and $a_n = C_2 \cdot (-4)^n$ are not consistent with the initial conditions. We conclude that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = -4$ (because $|-4| > |1|$).

Systems of recurrence equations

Example 60. Write the (second-order) RE $a_{n+2} = 4a_n - 3a_{n+1}$, with $a_0 = 1$, $a_1 = 2$, as a system of (first-order) recurrences.

Solution. (long form) Write $b_n = a_{n+1}$.

Then, $a_{n+2} = 4a_n - 3a_{n+1}$ translates into the first-order system $\begin{cases} a_{n+1} = b_n \\ b_{n+1} = 4a_n - 3b_n \end{cases}$.

Let $\boldsymbol{a}_n = \begin{bmatrix} a_n \\ b_n \end{bmatrix}$. Then, in matrix form, the RE is $\boldsymbol{a}_{n+1} = \begin{bmatrix} 0 & 1 \\ 4 & -3 \end{bmatrix} \boldsymbol{a}_n$, with $\boldsymbol{a}_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Solution. (short form) Equivalently, write $\boldsymbol{a}_n = \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix}$. Then we obtain the system

$$\boldsymbol{a}_{n+1} = \begin{bmatrix} a_{n+1} \\ a_{n+2} \end{bmatrix} = \begin{bmatrix} a_{n+1} \\ 4a_n - 3a_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 4 & -3 \end{bmatrix} \boldsymbol{a}_n, \quad \boldsymbol{a}_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Comment. It follows that $\boldsymbol{a}_n = \begin{bmatrix} 0 & 1 \\ 4 & -3 \end{bmatrix}^n \boldsymbol{a}_0 = \begin{bmatrix} 0 & 1 \\ 4 & -3 \end{bmatrix}^n \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Solving (systems of) REs is equivalent to computing powers of matrices!

Comment. We could also write $\boldsymbol{a}_n = \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$ (with the order of the entries reversed). In that case, the system is

$$\boldsymbol{a}_{n+1} = \begin{bmatrix} a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 4a_n - 3a_{n+1} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 1 & 0 \end{bmatrix} \boldsymbol{a}_n, \quad \boldsymbol{a}_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Comment. Recall that the **characteristic polynomial** of a matrix M is $\det(M - \lambda I)$. Compute the characteristic polynomial of both $M = \begin{bmatrix} 0 & 1 \\ 4 & -3 \end{bmatrix}$ and $M = \begin{bmatrix} -3 & 4 \\ 1 & 0 \end{bmatrix}$. In both cases, we get $\lambda^2 + 3\lambda - 4$, which matches the polynomial $p(N)$ (also called characteristic polynomial!) in the previous example. This will always happen and explains why both are referred to as the characteristic polynomial.

Example 61. Write $a_{n+3} - 4a_{n+2} + a_{n+1} + 6a_n = 0$ as a system of (first-order) recurrences.

Solution. Write $\boldsymbol{a}_n = \begin{bmatrix} a_n \\ a_{n+1} \\ a_{n+2} \end{bmatrix}$. Then we obtain the system

$$\boldsymbol{a}_{n+1} = \begin{bmatrix} a_{n+1} \\ a_{n+2} \\ a_{n+3} \end{bmatrix} = \begin{bmatrix} a_{n+1} \\ a_{n+2} \\ 4a_{n+2} - a_{n+1} - 6a_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -1 & 4 \end{bmatrix} \begin{bmatrix} a_n \\ a_{n+1} \\ a_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -1 & 4 \end{bmatrix} \boldsymbol{a}_n.$$

In summary, the RE in matrix form is $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$ with M the matrix above.

Important comment. Consequently, $\boldsymbol{a}_n = M^n \boldsymbol{a}_0$.

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Solving systems of recurrence equations

(systems of REs) The unique solution to $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$, $\boldsymbol{a}_0 = \boldsymbol{c}$ is $\boldsymbol{a}_n = M^n\boldsymbol{c}$.

We will say that M^n is the **fundamental matrix solution** to $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$ with $\boldsymbol{a}_0 = I$ (the identity matrix).

Comment. Note that $M^n\boldsymbol{c}$ is a linear combination of the columns of M^n. In other words, each column of M^n is a solution to $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$ and $M^n\boldsymbol{c}$ is the general solution.

In general, we say that a matrix Φ_n is a **matrix solution** to $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$ if $\Phi_{n+1} = M\Phi_n$. Φ_n being a matrix solution is equivalent to each column of Φ_n being a normal (vector) solution. If the general solution of $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$ can be obtained as the linear combination of the columns of Φ_n, then Φ_n is a **fundamental matrix solution**. Above, we observed that $\Phi_n = M^n$ is a special fundamental matrix solution.

If this is a bit too abstract at this point, have a look at Example 62 below.

However, at this point, it remains to figure out how to actually compute M^n. We will actually proceed the other way around: we construct the general solution of $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$ (see the box below) and then we use that to determine M^n from that.

To solve $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$, determine the eigenvectors of M.

- Each λ-eigenvector $\boldsymbol{v}$ provides a solution: $\boldsymbol{a}_n = \boldsymbol{v}\lambda^n$ [assuming that $\lambda \neq 0$]
- If there are enough eigenvectors, these combine to the general solution.

Why? If $\boldsymbol{a}_n = \boldsymbol{v}\lambda^n$ for a λ-eigenvector $\boldsymbol{v}$, then $\boldsymbol{a}_{n+1} = \boldsymbol{v}\lambda^{n+1}$ and $M\boldsymbol{a}_n = M\boldsymbol{v}\lambda^n = \lambda\boldsymbol{v}\cdot\lambda^n = \boldsymbol{v}\lambda^{n+1}$.

Where is this coming from? When solving single linear recurrences, we found that the basic solutions are of the form cr^n where $r \neq 0$ is a root of the characteristic polynomials. To solve $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$, it is therefore natural to look for solutions of the form $\boldsymbol{a}_n = \boldsymbol{c}r^n$ (where $\boldsymbol{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$). Note that $\boldsymbol{a}_{n+1} = \boldsymbol{c}r^{n+1} = r\boldsymbol{a}_n$.

Plugging into $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$ we find $\boldsymbol{c}r^{n+1} = M\boldsymbol{c}r^n$.

Cancelling r^n (just a nonzero number!), this simplifies to $r\boldsymbol{c} = M\boldsymbol{c}$.

In other words, $\boldsymbol{a}_n = \boldsymbol{c}r^n$ is a solution if and only if $\boldsymbol{c}$ is an r-eigenvector of M.

Comment. If there are not enough eigenvectors, then we know what to do as well (at least in principle): instead of looking only for solutions of the type $\boldsymbol{a}_n = \boldsymbol{v}\lambda^n$, we also need to look for solutions of the type $\boldsymbol{a}_n = (\boldsymbol{v}n + \boldsymbol{w})\lambda^n$. Note that this can only happen if an eigenvalue is a repeated root of the characteristic polynomial.

Example 62. Let $M=\begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix}$.

(a) Determine the general solution to $\boldsymbol{a}_{n+1}=M\boldsymbol{a}_n$.

(b) Determine a **fundamental matrix solution** to $\boldsymbol{a}_{n+1}=M\boldsymbol{a}_n$.

(c) Compute M^n.

Solution.

(a) Recall that each λ-eigenvector $\boldsymbol{v}$ of M provides us with a solution: $\boldsymbol{a}_n=\boldsymbol{v}\lambda^n$

We computed in Example 57 that $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda=3$, and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda=-2$.

Hence, the general solution is $C_1\begin{bmatrix} 2 \\ 1 \end{bmatrix}3^n+C_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}(-2)^n$.

(b) Note that we can write the general solution as

$\boldsymbol{a}_n=C_1\begin{bmatrix} 2 \\ 1 \end{bmatrix}3^n+C_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}(-2)^n=\begin{bmatrix} 2\cdot 3^n & (-2)^n \\ 3^n & (-2)^n \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$.

We call $\Phi_n=\begin{bmatrix} 2\cdot 3^n & (-2)^n \\ 3^n & (-2)^n \end{bmatrix}$ the corresponding **fundamental matrix (solution)**.

Note that our general solution is precisely $\Phi_n\boldsymbol{c}$ with $\boldsymbol{c}=\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$.

Observations.

(a) The columns of Φ_n are (independent) solutions of the system.

(b) Φ_n solves the RE itself: $\Phi_{n+1}=M\Phi_n$.
[Spell this out in this example! That Φ_n solves the RE follows from the definition of matrix multiplication.]

(c) It follows that $\Phi_n=M^n\Phi_0$. Equivalently, $\Phi_n\Phi_0^{-1}=M^n$. (See next part!)

(c) Note that $\Phi_0=\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, so that $\Phi_0^{-1}=\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$. It follows that

$$M^n=\Phi_n\Phi_0^{-1}=\begin{bmatrix} 2\cdot 3^n & (-2)^n \\ 3^n & (-2)^n \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}=\begin{bmatrix} 2\cdot 3^n-(-2)^n & -2\cdot 3^n+2(-2)^n \\ 3^n-(-2)^n & -3^n+2(-2)^n \end{bmatrix}.$$

Check. Let us verify the formula for M^n in the cases $n=0$ and $n=1$:

$M^0=\begin{bmatrix} 2-1 & -2+2 \\ 1-1 & -1+2 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$M^1=\begin{bmatrix} 2\cdot 3-(-2) & -2\cdot 3+2(-2) \\ 3-(-2) & -3+2(-2) \end{bmatrix}=\begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix}$

We just saw that being able to compute matrix powers is equivalent to solving systems of recurrences. Indeed, we can use this fact to compute matrix powers.

(a way to compute powers of a matrix M)
Compute a fundamental matrix solution Φ_n of $\boldsymbol{a}_{n+1}=M\boldsymbol{a}_n$.
Then $M^n=\Phi_n\Phi_0^{-1}$.

If you have taken linear algebra classes, you may have learned that matrix powers M^n can be computed by diagonalizing the matrix M. The latter hinges on computing eigenvalues and eigenvectors of M as well. Compare the two approaches!