Example 47. (cont'd) Let the sequence a_n be defined by $a_{n+2} = a_{n+1} + 6a_n$ and $a_0 = 1$, $a_1 = 8$.

(a) Determine the first few terms of the sequence.

(b) Find a formula for a_n.

(c) Determine $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

Solution.

(a) $a_2 = a_1 + 6a_0 = 14$, $a_3 = a_2 + 6a_1 = 62$, $a_4 = 146$, ...

(b) The recursion can be written as $p(N)a_n = 0$ where $p(N) = N^2 - N - 6$ has roots $3, -2$.
Hence, $a_n = C_1\, 3^n + C_2\, (-2)^n$ and we only need to figure out the two unknowns C_1, C_2. We can do that using the two initial conditions: $a_0 = C_1 + C_2 = 1$, $a_1 = 3C_1 - 2C_2 = 8$.
Solving, we find $C_1 = 2$ and $C_2 = -1$ so that, in conclusion, $a_n = 2 \cdot 3^n - (-2)^n$.
Comment. Such a formula is sometimes called a **Binet-like formula** (because it is of the same kind as the Binet formula for the Fibonacci numbers that we can derive in the same manner).

(c) It follows from our formula that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 3$ (because $|3| > |-2|$ so that 3^n dominates $(-2)^n$).
To see this, we need to realize that, for large n, 3^n is much larger than $(-2)^n$ so that we have $a_n \approx 2 \cdot 3^n$ when n is large. Hence, $\frac{a_{n+1}}{a_n} \approx \frac{2 \cdot 3^{n+1}}{2 \cdot 3^n} = 3$.
Alternatively, to be very precise, we can observe that (by dividing each term by 3^n)

$$\frac{a_{n+1}}{a_n} = \frac{2 \cdot 3^{n+1} - (-2)^{n+1}}{2 \cdot 3^n - (-2)^n} = \frac{2 \cdot 3 + 2\left(-\frac{2}{3}\right)^n}{2 \cdot 1 - \left(-\frac{2}{3}\right)^n} \quad \overset{\text{as } n\to\infty}{\longrightarrow} \quad \frac{2 \cdot 3 + 0}{2 \cdot 1 - 0} = 3.$$

Example 48. ("warmup") Find the general solution to the recursion $a_{n+2} = 4a_{n+1} - 4a_n$.

Solution. The recursion can be written as $p(N)a_n = 0$ where $p(N) = N^2 - 4N + 4$ has roots $2, 2$.
So a solution is 2^n and, from our discussion of DEs, it is probably not surprising that a second solution is $n \cdot 2^n$.
Hence, the general solution is $a_n = C_1 \cdot 2^n + C_2 \cdot n \cdot 2^n = (C_1 + C_2 n) \cdot 2^n$.
Comment. This is analogous to $(D-2)^2 y' = 0$ having the general solution $y(x) = (C_1 + C_2 x)e^{2x}$.
Check! Let's check that $a_n = n \cdot 2^n$ indeed satisfies the recursion $(N-2)^2 a_n = 0$.
$(N-2)n \cdot 2^n = (n+1)2^{n+1} - 2n \cdot 2^n = 2^{n+1}$, so that $(N-2)^2 n \cdot 2^n = (N-2)2^{n+1} = 0$.

Combined, we obtain the following analog of Theorem 24 for recurrence equations (RE):

Comment. Sequences that are solutions to such recurrences are called **constant recursive** or ***C*-finite**.

Theorem 49. Consider the homogeneous linear RE with constant coefficients $p(N)a_n = 0$.

- If r is a root of the characteristic polynomial and if k is its multiplicity, then k (independent) solutions of the RE are given by $n^j r^n$ for $j = 0, 1, \ldots, k-1$.
- Combining these solutions for all roots, gives the general solution.

Moreover. If r is the sole largest root by absolute value among the roots contributing to a_n, then $a_n \approx C r^n$ (if r is not repeated—what if it is?) for large n. In particular, it follows that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r.$$

Advanced comment. Things can get weird if there are several roots of the same absolute value. Consider, for instance, the case $a_n = 2^n + (-2)^n$. Can you see that, in this case, the limit $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ doesn't even exist?

Example 50. Find the general solution to the recursion $a_{n+3} = 2a_{n+2} + a_{n+1} - 2a_n$.

Solution. The recursion can be written as $p(N)a_n = 0$ where $p(N) = N^3 - 2N^2 - N + 2$ has roots $2, 1, -1$. (Here, we used some help from a computer algebra system to find the roots.)
Hence, the general solution is $a_n = C_1 \cdot 2^n + C_2 + C_3 \cdot (-1)^n$.

Example 51. Find the general solution to the recursion $a_{n+3} = 3a_{n+2} - 4a_n$.

Solution. The recursion can be written as $p(N)a_n = 0$ where $p(N) = N^3 - 3N^2 + 4$ has roots $2, 2, -1$. (Again, we used some help from a computer algebra system to find the roots.)
Hence, the general solution is $a_n = (C_1 + C_2 n) \cdot 2^n + C_3 \cdot (-1)^n$.

Theorem 52. (Binet's formula) $F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$

Proof. The recursion $F_{n+1} = F_n + F_{n-1}$ can be written as $p(N)a_n = 0$ where $p(N) = N^2 - N - 1$ has roots

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.618, \qquad \lambda_2 = \frac{1-\sqrt{5}}{2} \approx -0.618.$$

Hence, $F_n = C_1 \cdot \lambda_1^n + C_2 \cdot \lambda_2^n$ and we only need to figure out the two unknowns C_1, C_2. We can do that using the two initial conditions: $F_0 = C_1 + C_2 \overset{!}{=} 0$, $F_1 = C_1 \cdot \frac{1+\sqrt{5}}{2} + C_2 \cdot \frac{1-\sqrt{5}}{2} \overset{!}{=} 1$.
Solving, we find $C_1 = \frac{1}{\sqrt{5}}$ and $C_2 = -\frac{1}{\sqrt{5}}$ so that, in conclusion, $F_n = \frac{1}{\sqrt{5}}(\lambda_1^n - \lambda_2^n)$, as claimed. □

Comment. For large n, $F_n \approx \frac{1}{\sqrt{5}} \lambda_1^n$ (because λ_2^n becomes very small). In fact, $F_n = \operatorname{round}\left(\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n\right)$.

Back to the quotient of Fibonacci numbers. In particular, because λ_1^n dominates λ_2^n, it is now transparent that the ratios $\frac{F_{n+1}}{F_n}$ approach $\lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.618$. To be precise, note that

$$\frac{F_{n+1}}{F_n} = \frac{\frac{1}{\sqrt{5}}(\lambda_1^{n+1} - \lambda_2^{n+1})}{\frac{1}{\sqrt{5}}(\lambda_1^n - \lambda_2^n)} = \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1^n - \lambda_2^n} = \frac{\lambda_1 - \lambda_2\left(\frac{\lambda_2}{\lambda_1}\right)^n}{1 - \left(\frac{\lambda_2}{\lambda_1}\right)^n} \overset{n\to\infty}{\longrightarrow} \frac{\lambda_1 - 0}{1 - 0} = \lambda_1.$$

In fact, it follows from $\lambda_2 < 0$ that the ratios $\frac{F_{n+1}}{F_n}$ approach λ_1 in the alternating fashion that we observed numerically earlier. Can you see that?

Example 53. Consider the sequence a_n defined by $a_{n+2} = 4a_{n+1} + 9a_n$ and $a_0 = 1$, $a_1 = 2$. Determine $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

Solution. The recursion can be written as $p(N)a_n = 0$ where $p(N) = N^2 - 4N - 9$ has roots $\frac{4 \pm \sqrt{52}}{2} \approx 5.6056$, -1.6056. Both roots have to be involved in the solution in order to get integer values.
We conclude that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 2 + \sqrt{13} \approx 5.6056$ (because $|5.6056| > |-1.6056|$).

Example 54. (extra) Consider the sequence a_n defined by $a_{n+2} = 2a_{n+1} + 4a_n$ and $a_0 = 0$, $a_1 = 1$. Determine $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

First few terms of sequence. $0, 1, 2, 8, 24, 80, 256, 832, \ldots$
These are actually related to Fibonacci numbers. Indeed, $a_n = 2^{n-1} F_n$. Can you prove this directly from the recursions? Alternatively, this follows from comparing the Binet-like formulas.

Solution. Proceeding as in the previous example, we find $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1 + \sqrt{5} \approx 3.23607$.

Comment. With just a little more work, we find the Binet-like formula $a_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2\sqrt{5}}$.

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