Example 43. Suppose that a and b depend on x. Expand $(D+a)(D+b)$ in normal form.

Solution. $(D+a)(D+b)=D^2+Db+aD+ab=D^2+(bD+b')+aD+ab=D^2+(a+b)D+ab+b'$

Comment. Of course, if b is a constant, then $b'=0$ and we just get the familiar expansion.

Comment. At this point, it is not surprising that, in general, $(D+a)(D+b)\neq(D+b)(D+a)$.

Example 44. Suppose we want to factor D^2+pD+q as $(D+a)(D+b)$. [p,q,a,b depend on x]

(a) Spell out equations to find a and b.

(b) Find all factorizations of D^2. [An obvious one is $D^2=D\cdot D$ but there are others!]

Solution.

(a) Matching coefficients with $(D+a)(D+b)=D^2+(a+b)D+ab+b'$ (we expanded this in the previous example), we find that we need

$$p=a+b,\quad q=ab+b'.$$

Equivalently, $a=p-b$ and $q=(p-b)b+b'$. The latter is a nonlinear (!) DE for b. Once solved for b, we obtain a as $a=p-b$.

(b) This is the case $p=q=0$. The DE for b becomes $b'=b^2$.
Because it is separable (show all details!), we find that $b(x)=\frac{1}{C-x}$ or $b(x)=0$.

Since $a=-b$, we obtain the factorizations $D^2=\left(D-\frac{1}{C-x}\right)\left(D+\frac{1}{C-x}\right)$ and $D^2=D\cdot D$.

Our computations show that there are no further factorizations.

Comment. Note that this example illustrates that factorization of differential operators is not unique!
For instance, $D^2=D\cdot D$ and $D^2=\left(D+\frac{1}{x}\right)\cdot\left(D-\frac{1}{x}\right)$ (the case $C=0$ above).

Comment. In general, the nonlinear DE for b does not have any polynomial or rational solution (or, in fact, any solution that can be expressed in terms of functions that we are familiar with).

Solving linear recurrences with constant coefficients

Motivation: Fibonacci numbers

The numbers $0,1,1,2,3,5,8,13,21,34,\ldots$ are called **Fibonacci numbers**.

They are defined by the recursion $F_{n+1}=F_n+F_{n-1}$ and $F_0=0$, $F_1=1$.

How fast are they growing?
Have a look at ratios of Fibonacci numbers: $\frac{2}{1}=2$, $\frac{3}{2}=1.5$, $\frac{5}{3}\approx1.667$, $\frac{8}{5}=1.6$, $\frac{13}{8}=1.625$, $\frac{21}{13}=1.615$, $\frac{34}{21}=1.619$, ...

These ratios approach the **golden ratio** $\varphi=\frac{1+\sqrt{5}}{2}=1.618\ldots$

In other words, it appears that $\lim_{n\to\infty}\frac{F_{n+1}}{F_n}=\frac{1+\sqrt{5}}{2}$.

We will soon understand where this is coming from.

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We can derive all of that using the same ideas as in the case of linear differential equations. The crucial observation that we can write the recursion in operator form:

$F_{n+1} = F_n + F_{n-1}$ is equivalent to $(N^2 - N - 1)F_n = 0$.

Here, N is the shift operator: $Na_n = a_{n+1}$.

Comment. Recurrence equations are discrete analogs of differential equations.

For instance, recall that $f'(x) = \lim_{h \to 0} \frac{1}{h}[f(x+h) - f(x)]$.

Setting $h = 1$, we get the rough estimate $f'(x) \approx f(x+1) - f(x)$ so that D is (roughly) approximated by $N - 1$.

Example 45. Find the general solution to the recursion $a_{n+1} = 7a_n$.

Solution. Note that $a_n = 7a_{n-1} = 7 \cdot 7a_{n-2} = \cdots = 7^n a_0$.

Hence, the general solution is $a_n = C \cdot 7^n$.

Comment. This is analogous to $y' = 7y$ having the general solution $y(x) = Ce^{7x}$.

Solving recurrence equations

Example 46. ("warmup") Find the general solution to the recursion $a_{n+2} = a_{n+1} + 6a_n$.

Solution. The recursion can be written as $p(N)a_n = 0$ where $p(N) = N^2 - N - 6 = (N-3)(N+2)$.

Since $(N-3)a_n = 0$ has solution $a_n = C \cdot 3^n$, and since $(N+2)a_n = 0$ has solution $a_n = C \cdot (-2)^n$ (compare previous example), we conclude that the general solution is $a_n = C_1 \cdot 3^n + C_2 \cdot (-2)^n$.

Comment. This must indeed be the general solution, because the two degrees of freedom C_1, C_2 allow us to match any initial conditions $a_0 = A$, $a_1 = B$: the two equations $C_1 + C_2 = A$ and $3C_1 - 2C_2 = B$ in matrix form are $\begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix}$, which always has a (unique) solution because $\det\left(\begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix}\right) = -5 \neq 0$.

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